HOMEWORK QUESTIONS FOR MA 110B

HOMEWORK 1 (DUE BY FRIDAY FEB 3)

1. Suppose \( f \) is an entire function, and
\[
|f(z)| \leq A + B|z|^k,
\]
for every \( z \in \mathbb{C} \), where \( A, B \) and \( k \) are positive numbers. Prove that \( f \) is a polynomial.

2. Suppose \( f_n \) is a uniformly bounded sequence of holomorphic functions in a domain \( \Omega \) such that \( f_n(z) \) converges for every fixed \( z \in \Omega \). Prove that the convergence is uniform on every compact subset of \( \Omega \).

3. Let \( \gamma \) denote the positively oriented unit circle. Compute
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{e^z - e^{-z}}{z^4} \, dz.
\]

4. Suppose \( f_n \) are holomorphic functions on a domain \( \Omega \), and that none of the functions \( f_n \) has a zero in \( \Omega \). If \( f_n \) converges to a function \( f \) uniformly on compact sets in \( \Omega \), prove that either \( f \) has no zeroes or \( f(z) = 0 \) for every \( z \in \Omega \).

5. Suppose a domain \( \Omega \) contains the unit disc. Let \( f \) be holomorphic on \( \Omega \) and \( |f(z)| < 1 \) when \( |z| = 1 \). How many fixed points must \( f \) have in the unit disc (that is, how many solutions \( f(z) = z \) there are)?

6. Suppose \( f \) is holomorphic on \( \Omega \) and that \( \Omega \) contains the unit disc. Assume in addition that \( |f(z)| > 2 \) when \( |z| = 1 \) and \( f(0) = 1 \). Does \( f \) need to have a zero in the unit disc.

7. Denote by \( \mathbb{H} \) the upper half plane. Let \( f : \mathbb{H} \to \mathbb{C} \) be holomorphic and \( |f(z)| \leq 1 \) for \( z \in \mathbb{H} \). How large can \( |f'(i)| \) be. Find the extremal functions (namely those where \( |f'(i)| \) is maximized).
8. Suppose $\Omega$ is a bounded domain and $f_n$ a sequence of holomorphic functions defined on some neighborhood of $\Omega$. Prove that if $f_n$ converges uniformly on the boundary of $\Omega$, then $f_n$ converges uniformly on $\Omega$.

**Homework 2 (due by Friday Feb 17)**

1. Prove that if $f$ is holomorphic and has no zeroes on a domain $\Omega$, then the function $u(z) = \log |f(z)|$ is harmonic on $\Omega$.

2. Suppose that $u(z)$ is a positive harmonic function on the unit disc $\mathbb{D}$ and $u(0) = 1$. How small can $u\left(\frac{1}{2}\right)$ be? How large can $u\left(\frac{1}{2}\right)$ be?

3. Let $\Omega = \{z \in \mathbb{C} : \text{Re}(z) > 0\}$. Let $f$ be holomorphic on $\Omega$ and continuous on the closure $\overline{\Omega}$. Suppose that there are constants $A > 0$ and $0 < \alpha < 1$ such that

$$|f(z)| \leq Ae^{\alpha|z|},$$

for every $z \in \Omega$, and $|f(iy)| \leq 1$ for every $y \in \mathbb{R}$. Show $|f(z)| \leq 1$ for every $z \in \Omega$.

4. Let $f$ be holomorphic on the unit disc $\mathbb{D}$. Find a sequence $z_n \in \mathbb{D}$ such that

$$\lim_{n \to \infty} |z_n| = 1,$$

and

$$|f(z_n)| < M,$$

for every $n \in \mathbb{N}$ and some $M > 0$.

5. Is there a sequence of polynomials $P_n(z)$ such that

$$\lim_{n \to \infty} P_n(z) = \psi(z), \quad z \in \mathbb{C},$$

where $\psi(z) = 1$ for $\text{Re}(z) > 0$, $\psi(z) = -1$ for $\text{Re}(z) < 0$, and $\psi(z) = 0$ for $\text{Re}(z) = 0$. 
6. Let \( P_n(z) \) be a sequence of polynomials such that
\[
\lim_{n \to \infty} P_n(z) = \psi(z), \quad \text{for every } z \in \mathbb{C},
\]
for some function \( \psi : \mathbb{C} \to \mathbb{C} \). Show that \( \psi \) is holomorphic on an open and dense subset of \( \mathbb{C} \).

7. Let \( z_1, z_2 \in \mathbb{D} \) and set \( \Omega = \mathbb{D} \setminus \{z_1, z_2\} \). Describe the group \( \text{Aut}(\Omega) \).

8. Let \( f : \mathbb{D} \to \mathbb{D} \) be a Möbius transformation. Let
\[
f_n(z) = (f \circ \cdots \circ f)(z), \quad (n \text{ times}).
\]
Does the limit
\[
\lim_{n \to \infty} f_n(z),
\]
exist for every \( z \in \mathbb{D} \)? Characterize all such \( f \) so that the above limit exists for every \( z \in \mathbb{D} \).