HOMEWORK QUESTIONS FOR MA 110B

Homeowork 1 (due by Friday Feb 3)

1. Suppose \( f \) is an entire function, and

\[
|f(z)| \leq A + B|z|^k,
\]

for every \( z \in \mathbb{C} \), where \( A, B \) and \( k \) are positive numbers. Prove that \( f \) is a polynomial.

2. Suppose \( f_n \) is a uniformly bounded sequence of holomorphic functions in a domain \( \Omega \) such that \( f_n(z) \) converges for every fixed \( z \in \Omega \). Prove that the convergence is uniform on every compact subset of \( \Omega \).

3. Let \( \gamma \) denote the positively oriented unit circle. Compute

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{e^z - e^{-z}}{z^4} \, dz.
\]

4. Suppose \( f_n \) are holomorphic functions on a domain \( \Omega \), and that none of the functions \( f_n \) has a zero in \( \Omega \). If \( f_n \) converges to a function \( f \) uniformly on compact sets in \( \Omega \), prove that either \( f \) has no zeroes or \( f(z) = 0 \) for every \( z \in \Omega \).

5. Suppose a domain \( \Omega \) contains the unit disc. Let \( f \) be holomorphic on \( \Omega \) and \( |f(z)| < 1 \) when \( |z| = 1 \). How many fixed points must \( f \) have in the unit disc (that is, how many solutions \( f(z) = z \) there are)?

6. Suppose \( f \) is holomorphic on \( \Omega \) and that \( \Omega \) contains the unit disc. Assume in addition that \( |f(z)| > 2 \) when \( |z| = 1 \) and \( f(0) = 1 \). Does \( f \) need to have a zero in the unit disc.

7. Denote by \( \mathbb{H} \) the upper half plane. Let \( f : \mathbb{H} \to \mathbb{C} \) be holomorphic and \( |f(z)| \leq 1 \) for \( z \in \mathbb{H} \). How large can \( |f'(i)| \) be. Find the extremal functions (namely those where \( |f'(i)| \) is maximized).
8. Suppose $\Omega$ is a bounded domain and $f_n$ a sequence of holomorphic functions defined on some neighborhood of $\Omega$. Prove that if $f_n$ converges uniformly on the boundary of $\Omega$, then $f_n$ converges uniformly on $\overline{\Omega}$. 