Problem 7 solutions

2) (⇒) Suppose \( S \) is oriented.

⇒ \( \exists \) a unit normal vector field \( \mathbf{N} \) along \( \partial S \).

Then \( V = \{(p + t \mathbf{N}(p)) : t \in \mathbb{R}, p \in S \} \) for some \( \varepsilon > 0 \).

⇒ \( V' = \{(p + \varepsilon \mathbf{N}(p)) : p \in S \} \cup \{(p - \varepsilon \mathbf{N}(p)) : p \in S \} \) for some \( \varepsilon \in \mathbb{R} \).

\( S_i \) \( \Rightarrow \) \( S \), \( i \) differ onto its rings for \( i = 1, 2 \).

Let \( \psi : S \to \mathbb{R} \) be a diffeo onto its rings for \( i = 1, 2 \).

\( \psi \equiv \{(p + \varepsilon \mathbf{N}(p)) \}

Since \( \varepsilon < \varepsilon \), \( S_1 \cap S_2 = \emptyset \), so \( V' \) has two connected components, each of which is diffeo to \( S \).

(⇐) Suppose \( V' \) is not connected.

Claim: \( V' \) has exactly two connected components.

\( \mathcal{U} \): Let \( V_1 \) be a connected component of \( V' \).

Let \( f : V_1 \to \mathcal{C}(S) \)

\( (p + \varepsilon \mathbf{N}(p)) \mapsto (p) \).

We will first show that \( f \) is surjective. Suppose not. Let \( p_0, q_0 \in S \) s.t. \((p_0) \) lies in the image of \( f \), while \( (q_0) \) does not, and let \( \gamma : [0,1] \to S \) be a smooth path from \( p_0 \) to \( q_0 \). Let \( t_0 \in [0,1] \) s.t. \( \gamma(t_0) \notin \text{Im}(f) \) but \( \phi(1) \in \text{Im}(f) \) \( \forall 1 < 0 \).

(This exists \( \Rightarrow \) \( f \) is a local diffeo)

Let \( U \subseteq S \) be an orientable null of \( \gamma(t_0) \). Then for \( s < t \) and suff. close \( t \) to \( t_0 \), there is a smooth curve \( \eta : [s, t] \to U \) s.t. \( \eta(1) = \gamma(t_0) \).

Hence, \( \eta(1) = \gamma(t_0) \neq \eta(1) \) for some unit normal vector field \( N(x) \).
along \( \gamma(S) \). Let \( \gamma(t_0) := \gamma(t_0) \cdot (1 - N'\gamma(t_0)) \), and note that \( N'(\gamma(t_0)) = \nu'(t_0) \cdot \gamma'(t_0) \) is continuous. However, this contradicts \( \gamma(t_0) \in \text{Im}(f) \).

\( f \) is surjective.

Note that \( f: \mathcal{W} \to S \) is a 2:1-map.

Also, \( f \) restricted to each component of \( \mathcal{W} \) is surjective, so \( \mathcal{W} \) must have at most two connected components. Since \( \mathcal{W} \) is not connected, it has exactly two connected components. Since \( f \) restricted to each of these components is surjective, it is also injective on each of these components.

For \( \gamma \in S \), define \( N(\gamma) := (\gamma'(\gamma))^\tau \cdot (\gamma(\gamma)) - (\gamma(\gamma) \in \mathbb{R}^3). \) Note that \( N \) is a smooth normal vector field along \( \gamma(S) \), so \( S \) is orientable.

3) \( N(\gamma, \nu) = \frac{\left( \frac{\partial y}{\partial x} \right) \times \left( \frac{\partial v}{\partial x} \right)}{\sqrt{1 + \left( \frac{\partial y}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2}} \)

\[ = \left( \begin{array}{c} \frac{\partial y}{\partial x} \\ \frac{\partial v}{\partial x} \\ 0 \end{array} \right) \]

\[ \vdash \left( \frac{1}{2} N \right) \left( \gamma(\gamma) \right) = \left( \begin{array}{c} -s' \gamma(t) \\ 0 \\ 0 \end{array} \right) \bigg|_{(\gamma(\gamma), (\gamma(\gamma)) = (\gamma, \gamma)} = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \]

\[ \vdash L_{\gamma(0), \gamma(0)} \left( \frac{1}{2} N \right) \left( \gamma(0, 1) \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \]

\[ \vdash L_{\gamma(0), \gamma(0)} \left( \frac{1}{2} N \right) \left( \gamma(0, 1) \right) + b \left( \frac{1}{2} N \right) \left( \gamma(0, 1) \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \]
5) Choose local coordinates about \( y(0) \).

Then \( \langle q'(t), N(y(t)) \rangle = 0 \quad \forall t \in I. \)

\[ \Rightarrow \langle q''(t), N(y(t)) \rangle + \langle q'(t), L_{x(0)}(y(t)) \rangle = 0 \]

\[ \Rightarrow \Pi_{x(0)}(y(t)) = \langle q''(t), N(y(t)) \rangle. \quad \square \]

6) Suppose that \( q''(0) = 1\|q''(0)\| \cdot N(y(0)) \), where \( y(0) = y \).

Problem 5 \( \Rightarrow \quad |\Pi_{x(0)}(y'(0))| = |\langle q''(0), N(y(0)) \rangle| = \|q''(0)\| \)

\[ \therefore \text{It is sufficient to show that } q''(0) = 1\|q''(0)\| \cdot N(y(0)) \]

Since \( q \) lies in \( P \), so does \( q''(0) \).

\[ \langle q''(0), q'(0) \rangle = c \quad \text{for some constant } c. \]

\[ \Rightarrow \langle q''(0), q'(0) \rangle = 0. \]

\[ \therefore q''(0) \perp q'(0). \]

\[ \Rightarrow q''(0) = k \cdot N(y(0)) \quad \because N(y(0)) \in P \quad \text{and } \perp \quad q'(0). \]

\[ = \|q''(0)\| N(y(0)) \quad \because N(y(0)) \text{ is unit.} \quad \square \]