$\Delta_2$ is a union of triangles in $\Delta_i$.

We will prove that computing the Euler char. using $\Delta_i$ and $\Delta_2$ gives the same answer.

Consider a triangle $T$ of $\Delta_i$, which is a union of triangles $T_i, \ldots, T_k$ of $\Delta_2$. Suppose that there is a pair of triangles $T_i, T_j$ that share a common edge $e$, whose vertices do not both lie in $\delta T$. Then the collapse along the edge $e$ in the following procedure.

\[ \begin{array}{c}
\text{Note: } \\
\text{i) Collapsing does not change the vertices or edges on } \delta T. \\
\text{ii) Collapsing decreases } F \text{ by } 2 \\
\quad \text{E by } 3 \\
\quad \text{V by } 1 \\
\Rightarrow X(R) \text{ does not change under collapsing.}
\end{array} \]
Iteratively collapse the triangles in T until we cannot do so, i.e., every edge of the triangles in T have both vertices on \( \delta T \).

For any vertex \( v \) of the triangulation \( T \) that is not a vertex of \( \delta T \), choose an adjacent vertex \( u \) of the triangulation \( T \) along \( \delta T \). The slide from \( v \) to \( u \) in the following procedure.

If \( v, w \) are not adjacent vertices:

0/0

Note: The slide reduces \( H \) of vertices on \( \delta T \) by 1.

The slide reduces \( F \) by 1,

\( E \) by 2

\( V \) by 1

\( \Rightarrow \chi(R) \) does not change.
Iterate until the triangulation $\Delta_i$ becomes $\Delta_j$.

**Theorem:** (Classification of compact, orientable smooth surfaces)

1. If $S_1, S_2$ are two compact, orientable smooth surfaces, then $\chi(S_1) = \chi(S_2)$, then $S_1$ is diffeomorphic to $S_2$.

2. If $S$ is a compact, orientable surface, then $\chi(S) = 2$ and $\chi(S)$ is even.

\[ \chi(S) = 2, \quad \chi(S) = 0, \quad \chi(S) = 2 - 2g. \]

**Index of vector fields**

$q: S \to \mathbb{R}^3$ a smooth immersion.

Let $V$ be a smooth unit vector field along $S$.

**Def.** $p \in S$ is a **regular point** of $V$ if $V(p) \neq 0$.

A regular point $p$ of $V$ is **isolated** if $\exists$ open $U \subseteq S$ containing $p$ s.t. $V[p] \subseteq U \setminus \{p\}$.

**Note.** If $S$ is compact, $V$ has isolated singular points $\iff$ $V$ has finitely many singular points.

Choose an orientation on $S$.

Let $V: [a,b] \to S$ be a simple, closed, properly oriented smooth curve s.t.

the region $R$ bounded by $S$ is simply connected and lies in a chart $\psi$ of $S$ s.t. $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle = 0$ for $p$ in the chart.

Let $V$ be a vector field along $S$. 
Recall: \( \exists \bar{\theta} : [a, b] \rightarrow \mathbb{R} \) smooth s.t. \( V(\bar{\theta}(t)) = \cos(\bar{\theta}(t)) \cdot \frac{1}{3X} \bar{L} + \sin(\bar{\theta}(t)) \cdot \frac{1}{3Y} \bar{L} \)

\( \bar{\theta}(b) - \bar{\theta}(a) \)

\[
\left[ \frac{dV}{dt} \right] = \frac{1}{2 \sqrt{EG}} \left( \frac{1}{3X} \bar{x}'(t) - \frac{1}{3Y} \bar{y}'(t) \right) + \bar{\theta}'
\]

Then \( E = \left\langle \frac{1}{3X}, \frac{1}{3Y} \right\rangle \)

\( G = \left\langle \frac{1}{3Y}, \frac{1}{3Y} \right\rangle \)

\( \Rightarrow \) If \( V \) is parallel along \( \bar{\theta} \), then

\[
0 = \int_a^b \frac{dV}{dt} \, dt
\]

\[
= \int_a^b \frac{1}{2 \sqrt{EG}} \left( \frac{1}{3X} \bar{x}'(t) - \frac{1}{3Y} \bar{y}'(t) \right) \, dt + \int_a^b \bar{\theta}' \, dt
\]

\( \text{hert Gauss} \quad \Rightarrow \quad \int_R \bar{K} \, d\sigma = \bar{\theta}(b) - \bar{\theta}(a) \)

\( \text{on bord} \quad \Rightarrow \quad \int_R \bar{K} \, d\sigma = \bar{\theta}(b) - \bar{\theta}(a) \).
Prop: The index of \( V \) at \( p \) does not depend on the choice of \( \gamma \), i.e., if \( \gamma : [a, b] \to S \) is another simple, closed, locally oriented smooth curve that bounds a simply connected region \( R' \) in a chart of \( S \) s.t. \( p \in R' \), then the index computed using \( \gamma \) is equal to the index computed using \( \gamma_1 \).

Ps.: Let \( H : [0, 1] \times [a, b] \to S \) be a smooth map s.t. \( H(0, t) = \gamma(t) \), \( H(1, t) = \gamma_1(t) \), and \( H_s(t) = H(s, t) \) in a smooth, closed curve. (\( H \) is an isotopy between \( \gamma \) and \( \gamma_1 \)).

Then we view \( I \) as a function \( I : [0, 1] \to \mathbb{R} \).

\[ I(s) = \text{index of } H_s. \]

I is continuous, but is integer valued.

\[ \Rightarrow I \text{ is constant.} \]

Prop: If \( p \) is non-singular, then the index at \( p \) is 0.

Ps.: Fact: \( \exists \) chart \( U \ni p \) s.t. \( \frac{\partial}{\partial x} |_p = V(p) \)

\[ \Rightarrow \theta(b) = 0 = \theta(a). \]

Example: 1)
Theorem: (Poincaré) If $S$ is compact and $V$ has only isolated singular points, then the sum of the indices at the singular points is $\chi(S)$.

Pf: Let $T$ be a finite triangulation of $S$ s.t.

i) each $T \in T$ lies in a chart s.t. $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle = 0$.

ii) each $T \in T$ contains at most one singular point.

iii) the boundary of each $T \in T$ does not contain singular points, and is truly oriented.

For each $T \in T$, $\int_T K \omega - 2\pi I_T = \int_T \phi'(t) \, dt - \int_T \phi'(t) \, dt$. For a smooth angle field.
\[ \Rightarrow \oint_{\Sigma} \sum_{T \epsilon \mathcal{T}} T \cdot A = \frac{1}{2\pi} \oint_{S} K \, d\sigma = \chi(S). \]

Cor: Any vector field on \( S^2 \) must have a singular point.

Back to geodesics. Let \( g: S \to \mathbb{R}^3 \) be an immersion.

Prop: \( \forall p \in S, \ \forall T_p \in T_p S, \ \exists \gamma \in C^1 \) s.t. \( \gamma: (-\epsilon,\epsilon) \to S \) is a geodesic \( \gamma(0) = p, \ \gamma'(0) = T_p \).

Furthermore, if \( \gamma \) is another such geodesic, then \( \gamma = \gamma'. \)

If: Choose local coordinates about \( p \), \( \gamma(t) = (x(t), y(t), z(t)) \) is a geodesic,

\[ \Rightarrow \frac{d}{dt} = \frac{D(x')}{\text{old}} \]

\[ = \frac{1}{\sqrt{g}} \left( \sum_{\nu=1}^{2} \left( \gamma_{\nu}'(t) + \sum_{i,j} \Gamma_{\nu i}^{k} \gamma_{i}'(t) \gamma_{j}'(t) \right) \right) \frac{1}{\sqrt{g}}. \]

\[ \left\{ \begin{array}{l} \gamma_{\nu}'(t) + \sum_{i,j} \Gamma_{\nu i}^{k} \gamma_{i}'(t) \gamma_{j}'(t) = 0 \\ \gamma_{\nu}''(t) + \sum_{i,j} \Gamma_{\nu i}^{k} \gamma_{i}'(t) \gamma_{j}'(t) = 0 \end{array} \right\} \]

has a unique solution locally.

Let \( \gamma_{p,v} \) denote the geodesic \( w/ \ \gamma_{p,v}'(0) = p, \ \gamma_{p,v}(0) = v \).
Lemma: If $\gamma_{p,v} : (-\varepsilon, \varepsilon) \to S$ then $\gamma_{p,v}^{'1} : (-\delta, \delta) \to S$ defined by $\gamma_{p,v}(t) : = \gamma_{p,v}(t \varepsilon)$ is the geodesic $\gamma_{p,v}$.

Proof: $\gamma_{p,v}(0) = \gamma_{p,v}(0) = p;\gamma_{p,v}(0)' = \gamma_{p,v}(0): A = \lambda v$.

Theorem: If $p \in S$, $\exists U_p \supseteq TpS$ containing $0$ s.t. exp: $U_p \to S$ in well-defined.

Proof: Consider the vector field on $TS$, i.e. $\sigma : TS \to T(TS)$

$\gamma_{p,v} : (-\varepsilon, \varepsilon) \to TS$ given by $X(p,v) = \gamma_{p,v}(0)$, where

$$\gamma_{p,v} : (-\varepsilon, \varepsilon) \to TS, \gamma_{p,v}(t) \to (\gamma_{p,v}(t), \gamma_{p,v}'(t))$$

Lemma: If $X$ is a smooth vector field on an open set $U \subseteq \mathbb{R}^n$ and $q \in U$ then $\exists \gamma_q : (-\varepsilon, \varepsilon) \to U$ s.t. $\gamma_q : (-\varepsilon, \varepsilon) \to U$ is a smooth map $\gamma : (-\varepsilon, \varepsilon) \times U \to U$ s.t. $\gamma_q : (-\varepsilon, \varepsilon) \to U$ satisfies $\gamma_q(0) = q$, $\gamma_q'(0) = X(\gamma_q(0))$.

Applying this lemma to $X$ with $U$ a chart of $TM$, $q = (q, 0)$, by shrinking $U$, if necessary, can assume $\forall V_0 \subseteq \partial S, p \in V_0 \Rightarrow U_0 : \mathbb{R}^2 \cap \mathbb{R}^3 = \{(a, v) \in TS : a \in V_0, |v| < \varepsilon\}$

for some $\varepsilon > 0$.

In particular, $\mathbb{R}^2 \subseteq \mathbb{R}^3 \times \mathbb{R}^3 : \mathbb{R}^2 \cap \mathbb{R}^3 \cap \mathbb{R}^3 \to U \subseteq TS$ is smooth and $\gamma_{p,v} : (-\varepsilon, \varepsilon) \to TS$ satisfies $\gamma_{p,v}(0) = (p, 0)$.

Let $T : T_pS \to S$, then $\gamma_{p,v}(t) = T(\gamma_{p,v}(t))$. 
Lemma: If $\gamma_{p,v} : (-\varepsilon, \varepsilon) \to S$ is a geodesic, then $\gamma_{p,v} : (-\varepsilon, \varepsilon) \to S$ is in the geodesic $\gamma_{p,v}$, $\forall \varepsilon > 0$.

Proof: 
\[ \frac{d}{dt}(\gamma_{p,v}(t)) = \frac{d}{dt}(\gamma_{p,v}(t)) \]
\[ = \frac{d}{dt}(\gamma_{p,v}(0)) \cdot \frac{1}{\varepsilon}. \]
\[ = 0. \quad \therefore \gamma_{p,v} \text{ is a geodesic}. \]

\[ \Rightarrow \gamma_{p,v} \text{ is a geodesic}. \]

\[ \gamma_{p,v}'(0) = \gamma_{p,v}'(0) \cdot \frac{1}{\varepsilon}. \]
\[ \Rightarrow \gamma_{p,v}'(0) = \frac{1}{\varepsilon} v. \]

\[ \gamma_{p,v}(0) = p. \]
\[ \Rightarrow \gamma_{p,v} = \gamma_{p,v}. \]

\[ \Rightarrow \forall v \in T_p S, \exists \varepsilon > 0, \gamma_{p,v} : (-\varepsilon, \varepsilon) \to S \text{ is a geodesic}. \]

Let $U_p := \gamma_{p,v}^{-1}(T_p S) : \forall v \leq \varepsilon$.

Prop: $\forall p \in S$, $\exists U_{p,v}, T_p S$ s.t. $U > 0$ and $\exp_p |_{U}$ is a diffeomorphism.

Proof: 
\[ \exp : T_p S \to S, \quad (\exp)_p : T_0(T_p S) \to T_p S. \]
\[ (\exp)_p(v) = \frac{d}{dt}|_{t=0} \exp_p(tv) \]
\[ = \gamma_{p,v}'(0) \]
\[ = v. \]

\[ \therefore (\exp)_p|_{U} = \text{id}, \quad \text{so IFT} \Rightarrow \exp_p \text{ is a local diffeomorphism about } 0. \]
Example: i) \( S = \mathbb{R}^2, \ p \in \mathbb{R}^2 \).

\[
\exp_p : T_p \mathbb{R}^2 \cong \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \text{ in a global diffeomorphism.}
\]

\[
v \longrightarrow p + v.
\]

ii) \( S = \mathbb{S}^2, \ p = \text{north pole} \).

\[
\exp_p : T_p \mathbb{S}^2 \longrightarrow \mathbb{S}^2, \text{ in a diffeo on } B_{\pi}(0).
\]

iii) \( S = \mathbb{D}^2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \).

\[
\exp_p : B_1(0) \longrightarrow \mathbb{D}^2.
\]

\[
\frac{G}{T_0 \mathbb{D}^2} \longrightarrow v.
\]