Pushing forward vectors

Let \( M, M_2 \) be smooth manifolds.

**Def:** Let \( \varphi : M \to M_2 \) be a smooth map and let \( p \in M_1 \). The pushforward of \( \varphi \) in the map \( \varphi* : T_p M \to T_{\varphi(p)} M_2 \), where \( \varphi* (X)(f) = X(\varphi \circ f) \). \( \forall X \in T_p M, \forall f \in C^\infty(M) \).

**Check:**
- \( \varphi* (X) \) is a derivation.
- \( \varphi* \) is a linear map.

**Prop:** Let \( \varphi_1 : M \to M_1 \) and \( \varphi_2 : M_2 \to M_3 \) be smooth maps and \( p \in M \). Then \( (\varphi_2 \circ \varphi_1)*_p = (\varphi_2)*_{\varphi_1(p)} \circ (\varphi_1)*_p : T_p M \to T_{(\varphi_2 \circ \varphi_1)(p)} M_3 \).

**Pf:** Let \( f \in C^\infty(M_3) \), \( X \in T_p M \).

\[
(\varphi_2 \circ \varphi_1)*_p (X)(f) = X((\varphi_2 \circ \varphi_1)(f))
\]

\[
(\varphi_2)*_{\varphi_1(p)} \circ (\varphi_1)*_p (X)(f) = (\varphi_1)*_p (X)((\varphi_2 \circ \varphi_1)(f))
\]

\[
= X((\varphi_2 \circ \varphi_1)(f)).
\]

**Cor:** If \( \varphi : M \to M_2 \) is a diffeomorphism, then \( \forall p \in M, \varphi*: T_p M \to T_{\varphi(p)} M_2 \) is an isomorphism, and \( \varphi^{-1}*_{p} = (\varphi^{-1})*_{\varphi(p)}. \)

**Pf:** \( \varphi \circ \varphi^{-1} = \text{id}_M : M \to M \),

\[
\Rightarrow (\varphi^{-1})*_{\varphi(p)} \circ \varphi*_{p} = \text{id}_{T_p M} : T_p M \to T_p M,
\]

Similarly, \( \varphi*_{p} \circ (\varphi^{-1})*_{\varphi(p)} = \text{id}_{T_{\varphi(p)} M_2} : T_{\varphi(p)} M_2 \to T_{\varphi(p)} M_2 \).

**Def:** A smooth map \( \varphi : M \to M_2 \) is a smooth immersion if \( \varphi* \) is injective \( \forall p \in M \).

Diffeomorphisms are smooth immersions.
Let \( \gamma: (-\epsilon, \epsilon) \to M \) be smooth and \((U_{\alpha}, \gamma_{\alpha})\) a smooth chart of \(M\) s.t. \(\gamma(t_0) \in U_{\alpha}\) for some \(t_0 \in (-\epsilon, \epsilon)\). Then

\[
\frac{d((\gamma_{\alpha} \circ \gamma)^{-1}(t_0))}{dt}
= ((\gamma_{\alpha} \circ \gamma)^{-1})_{t_0}(\frac{1}{t_0} \big|_{t_0})
= \left((\gamma_{\alpha}^{-1})_{t_0(\gamma(t_0))} \circ \gamma_{\alpha}(t_0)\right)^{-1}(\frac{1}{t_0} \big|_{t_0}).
\]

Since \(\gamma_{\alpha}\) is a local diffeomorphism, \((\gamma_{\alpha} \circ \gamma)^{-1}\) is injective.

\[
\therefore \gamma \text{ is a smooth curve } \iff \gamma_{\alpha}(t_0) \left(\frac{1}{t_0} \big|_{t_0}\right) \not\equiv 0 \quad \forall \ t_0 \in (-\epsilon, \epsilon)
\]

\[
\gamma'(t_0).
\]

Cor: Let \(\gamma: (-\epsilon, \epsilon) \to M_1\) be a smooth curve and \(\varphi: M_1 \to M_2\) a smooth immersion. Then \(\varphi \circ \gamma: (-\epsilon, \epsilon) \to M_2\) is a smooth curve, and \(\forall t_0 \in (-\epsilon, \epsilon)\)

\[
(\varphi \circ \gamma)'(t_0) = \varphi_{\gamma(t_0)}'(\gamma'(t_0)).
\]

Pf: \(\gamma\) is a smooth curve \(\Rightarrow \gamma'(t_0) \not\equiv 0\).

\(\varphi\) is a smooth immersion \(\Rightarrow \varphi_{\gamma(t_0)}'(\gamma'(t_0)) \not\equiv 0\).

Previous corollary \(\Rightarrow 0 \neq \varphi_{\gamma(t_0)}'(\gamma_{\alpha}(t_0) \left(\frac{1}{t_0} \big|_{t_0}\right))
\]

\[
= (\varphi \circ \gamma)'_{t_0}(\frac{1}{t_0} \big|_{t_0})
= (\varphi \circ \gamma)'(t_0).
\]

Example: Is \(\varphi: \mathbb{R}^2 \to \mathbb{R}^2\)

\[(x, y) \mapsto (x, y, x^2 + y^2)\]

a smooth immersion?

Choose any \(p = (x_0, y_0) \in \mathbb{R}^2\), then \(\frac{1}{2x} \big|_p, \frac{1}{2y} \big|_p, \frac{1}{2} \big|_p\) is a basis of \(T_p \mathbb{R}^2\).

\(\therefore\) Need to check \(\varphi_{\star p}(\frac{1}{2x} \big|_p)\) and \(\varphi_{\star p}(\frac{1}{2y} \big|_p)\) are linearly independent.

\[
\varphi_{\star p}(\frac{1}{2x} \big|_p)(x_0, 0, y_0) = \frac{1}{2x} \big|_p (x_0, 0, y_0) = 1.
\]
\[ q_{*,p} \left( \frac{1}{3x} \middle|_p \right) (x) = \frac{1}{3x} \left|_{\varepsilon_{(p)}} \right. 
\]
\[ = 0. \]
\[ q_{*,p} \left( \frac{1}{3x} \middle|_p \right) (x) = \frac{1}{3x} \left|_{\varepsilon_{(p)}} \right. \]
\[ = 2x_0. \]
\[ \therefore q_{*,p} \left( \frac{1}{3x} \middle|_p \right) = \frac{1}{3x_0} \varepsilon_{(p)} + 0 \cdot \frac{1}{3x_0} \varepsilon_{(p)} + 2x_0 \cdot \frac{1}{3x_0} \varepsilon_{(p)} \]
\[ = \frac{1}{3x_0} \varepsilon_{(p)}. \]

Similarly, \[ q_{*,p} \left( \frac{1}{3y} \middle|_p \right) = \left( \frac{1}{3y_0} \right) \]
\[ \therefore q_{*,p} \left( \frac{1}{3x} \middle|_p \right), q_{*,p} \left( \frac{1}{3y} \middle|_p \right) \text{ are linearly independent \& } p \in \mathbb{R}^2. \]

\[ \Rightarrow \varphi \text{ is an immersion.} \]

Is \[ \varphi: \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \]
\[ (x,y) \longmapsto (xy+2y, x^2+4y, x^2) \]

Let \[ p=(x_0,y_0). \] Then \[ q_{*,p} \left( \frac{1}{3x} \middle|_p \right) = \left( \frac{y_0}{2x_0}, \frac{y_0}{2x_0} \right), \quad q_{*,p} \left( \frac{1}{3y} \middle|_p \right) = \left( \frac{x_0}{2y_0}, -\frac{x_0}{2y_0} \right) \]

When \[ p=(0,2), \] then \[ q_{*,p} \left( \frac{1}{3x} \middle|_p \right) = \left( \frac{2}{0}, 0 \right) = q_{*,p} \left( \frac{1}{3y} \middle|_p \right) \]

\[ \therefore \varphi \text{ is not an immersion at } (0,2). \]

Exercise: Show that \[ \varphi: \mathbb{S}^2 \longrightarrow \mathbb{R}^3 \] is not a smooth immersion.
\[ (x,y) \longmapsto (x,y) \]

Show that \[ \text{id}: \mathbb{S}^2 \longrightarrow \mathbb{R}^3 \] is a smooth immersion.
\[ (x,y) \longmapsto (x,y) \]

Here, smooth structure on \[ \mathbb{S}^2 \] given by charts defined in 1st lecture.

When \[ \varphi: \mathbb{S} \longrightarrow \mathbb{R}^3 \] is a smooth immersion, we can think of \[ T_p\mathbb{S} \] as \[ (q_{*,p})(T_p\mathbb{S}) \in T_{q_{*,p}} \mathbb{R}^3. \] This can be described using smooth maps.
Example: Let $S = S^2$, $q = \text{id}: S^3 \to \mathbb{R}^3$, $p = (x_0, y_0, z_0) \in S^2$.

\[ q(\bar{r}, \sin \bar{q}, \cos \bar{q}) = (\cos \bar{r} \sin \bar{q}, \sin \bar{r} \sin \bar{q}, 1) \]

Let $Y_1(\bar{r}) = (\cos \bar{r} \sin \bar{q}, \sin \bar{r} \sin \bar{q}, \cos \bar{q})$,

$Y_2(\bar{r}) = (\cos \bar{r} \sin \bar{q}, \sin \bar{r} \sin \bar{q}, 1)$. 

Check: When $\bar{q} \neq k\pi$ for some $k \in \mathbb{Z}$, then $Y_1$ and $Y_2$ are smooth curves.

\[ Y_1(\bar{r}) = Y_2(\bar{q}) = p. \]

\[ Y_1'(\bar{r}) = \begin{pmatrix} -\sin \bar{r} \sin \bar{q} \\ \cos \bar{r} \cos \bar{q} \\ 0 \end{pmatrix}, \quad Y_2'(\bar{q}) = \begin{pmatrix} \cos \bar{r} \sin \bar{q} \\ \sin \bar{r} \sin \bar{q} \\ 0 \end{pmatrix} \]

If $Y_1'(\bar{r})$ and $Y_2'(\bar{q})$ are not linearly independent, then $\sin \bar{q} = 0$.

\[ \Rightarrow \bar{q} = k\pi \text{ for } k \in \mathbb{Z}. \]

If $p$ is not one of the two poles, then

\[ \Sigma_2 (T_p S^2) = \{ aY_1'(\bar{r}) + bY_2'(\bar{q}) : a, b \in \mathbb{R} \} = \{ (\frac{x}{y}, \frac{y}{y}, \frac{z}{y}) : x, y, z \in \mathbb{R} \}. \]

Partition of unity

If there is a family of smooth functions defined on each chart of a smooth surface $S$, how can one "patch them together" to get a smooth function on $S$? Partition of unity is a tool to do so.

Definition: Let $U_1, U_2$ be open covers of $S$. $U_2$ is a refinement of $U_1$ if $\forall U \in U_1, \exists V \in U_2$ s.t. $V \subseteq U$. 
Theorem: Let \( U \) be an open cover of \( S \). Then \( U \) has a refinement \( U_2 \) s.t.

1) \( U_2 \) is locally finite, i.e. \( \forall p \in S, \exists U \in U_2 : p \in U \setminus \text{ is finite.} \)
2) \( U_2 \) is countable.
3) \( \forall U \in U_2, \exists \) smooth function \( f_U : S \rightarrow \mathbb{R} \) s.t. \( f_U(p) = 0 \ \forall p \in S \setminus U \)
   and \( \sum_{U \in U_2} f_U(p) = 1 \ \forall p \in S. \)
4) \( \forall U \in U_2, \overline{U} \subseteq S \) is compact. (\( U_2 \) is a precompact cover).

Fact: If \( X \) is a 2nd countable space, then every open cover has a countable subcover.

Lemma: \( S \) admits a countable, locally finite, precompact cover \( U = \{ V_i \}_{i=1}^\infty. \)

Pf: \( U = \{ V_i \}_{i=1}^\infty. \) is a cover of \( S. \)

Let \( B(U_i) = \{ \tilde{t}_i (B_r(p)) : B_r(p) \subseteq \tilde{t}_i(U_i), \ r \in \mathbb{Q}, \ p \in \mathbb{R}^n \} \)

Let \( B = \bigcup_{i=1}^\infty B(U_i). \)

Note that \( B \) is countable and \( \forall B \in B, \overline{B} \) is compact.

Let \( B = \{ B_i \}_{i=1}^\infty. \)
Define an intermediate cover \( \{ \overline{W_i} \}_i \) of \( S \) s.t.

1) \( \overline{W_i} \subseteq S \) is compact
2) \( \overline{W_i} \subseteq W_{i+1} \) \( \forall i \)
3) \( B_i \subseteq \overline{W_i} \).

We will do this inductively.

Let \( W_1 := B_1 \).

Suppose \( W_1, \ldots, W_k \) are open sets satisfying (i) - (iii).

Since \( \overline{W_k} \) is compact, \( \exists m_k \in \mathbb{Z}^+ \) s.t. \( \overline{W_k} \subseteq B_1 \cup \cdots \cup B_{m_k} \).

Can assume \( m_k \geq k + 1 \).

Then let \( W_{k+1} := \bigcup_{i=1}^{m_k} B_i \), and claim (i) - (iii) hold for \( W_1, \ldots, W_{k+1} \).

Finally, define \( V_1 := W_1, V_2 := W_2, V_k := W_k \setminus \overline{W_{k-2}} \) for \( k \geq 3 \).

Each \( \overline{V_k} \) is compact, \( \overline{V_k} \subseteq \overline{W_k} \).

Also, \( V_k \cap V_2 \neq \emptyset \iff k = 1 \) or \( k + 1 \).

**Proof.**

Let \( p \in S \), let \( U \in U_1 \) be an open set containing \( p \) and \( U_0 \) a chart of \( S \) containing \( p \).

Let \( W_p := U \cap U_0 \cap \bigcap_{i=p}^n V_i \neq \emptyset \).

Let \( t_p := t_{U_0}|_{W_p} \).

Note that \( t_p(W_p) \subset \mathbb{R}^2 \) contains a ball \( B_{r_0}(t_p(p)) \) for some \( r_0 > 0 \).

Let \( W_p := t_p^{-1}(B_{3r}(t_p(p))) \)

\( X_p := t_p^{-1}(B_r(t_p(p))) \).

\( \forall k, \overline{V_k} \) is compact \( \overline{V_k} \), so \( \exists \{ X_{k,i} \} \) s.t. \( \overline{V_k} \subseteq \bigcup_{i=1}^{n_k} X_{k,i} \).
If $x_{k,i} = x_i$ for some $p \in S$, let $W_k := W_p$.

Let $t_{k,i} := t_p |_{W_{x_{k,i}}}$.

Note that $\{W_{x_{k,i}}\} = U_2$.

We now argue that $U_2$ is locally finite.

Let $V_k \in V$, let $V_k := \{V \in V : V \cap V_k \neq \emptyset\}$

$\{V_k, \ldots, V_k\}$

$V$ is locally finite $\Rightarrow$ $V_k$ is finite.

Suppose $V_{k_0} \cap V_k$. Then $V_{k_0} \cap V_{k_0} \cap W_{x_{k_0},i}$ does not intersect $V_k$.

The only elements in $U_2$ that intersect $V_k$ are of the form $W_{x_{k,i}}$ for $i = 1, \ldots, n$, and then are finitely many of them.

$\Rightarrow$ Any point in $V_k$ lies in finitely many open sets in $U_2$.

$k$ is arbitrary, so $U_2$ is locally finite.

For $k, V_i = 1, \ldots, l_k$, let $h_{x_{k,i}} : IR^2 \rightarrow IR$ be a smooth function s.t.

$h_{x_{k,i}}(q) = 0$ if $q \notin B_r(t_{x_{k,i}}(p))$ and $h_{x_{k,i}}(p) = 1$ if $q \in B_r'(t_{x_{k,i}}(p))$.

Here, $p$ is the point s.t. $W_{x_{k,i}} = W_p$. 

\[ W_{x_{k,i}} := W_p \]
Then let \( f_{k,i} : S \rightarrow \mathbb{R} \) be given by \[
    f_{k,i}(g) = \begin{cases} 
    0 & \text{if } g \in \mathcal{W}_{k,i} \\
    \frac{g_{k,i}(g)}{\sum_{k,i} g_{k,i}(g)} & \text{if } g \notin \mathcal{W}_{k,i} 
    \end{cases}
\]

\[ f_{k,i} : S \rightarrow \mathbb{R}. \]

\[ g \mapsto \frac{g_{k,i}(g)}{\sum_{k,i} g_{k,i}(g)} \]

Note: \[ \sum_{k,i} g_{k,i}(g) \] is defined \[ \Rightarrow \mathcal{U}_k \text{ is locally finite.} \]

\[ \sum_{k,i} g_{k,i}(g) > 0 \Rightarrow \mathcal{U}_k \text{ is a cover.} \]

\[ f_{k,i} \text{ is smooth} \Rightarrow \text{each } g_{k,i} \text{ is smooth.} \]

\[ f_{k,i}(g) = 0 \quad \forall \ g \in S \setminus \mathcal{W}_{k,i}. \]

\[ \sum_{k,i} f_{k,i}(g) = \frac{\sum_{k,i} g_{k,i}(g)}{\sum_{k,i} g_{k,i}(g)} = 1 \quad \forall \ g \in S. \]

**The tangent bundle**

**Def:** The tangent bundle of \( S \), \( TS := \{(p, X) : p \in S, X \in T_pS\} \)

\[ \bigcup_{p \in S} T_pS. \]

**Goal:** Specify charts on \( TS \) to make it a smooth manifold.

**Observe:** If \( U \subseteq \mathbb{R}^n \) is open, then \( F_U : TU \rightarrow U \times \mathbb{R}^2 \) is a bijection, and \( U \times \mathbb{R}^2 \subseteq \mathbb{R}^n \) is open.

\( \Rightarrow TU \) is a smooth manifold by precomposing charts of \( U \times \mathbb{R}^2 \) by \( F_U \).
Let \((U_\alpha, \tau_\alpha)\) be a chart of \(S\), and let \(U \equiv \tau_\alpha(U_\alpha) \subseteq \mathbb{R}^n\).

Then \(G_\alpha: TU_\alpha \rightarrow TU\) is a bijection.
\((p, X) \mapsto (\tau_\alpha(p), (\tau_\alpha)_* p(X))\)

\(\Rightarrow\) each \(TU_\alpha\) is a smooth manifold.

\(TS = \bigcup_\alpha TU_\alpha\).

Equip \(TS\) with the topology generated by the topologies on \(TU_\alpha\).

Check: The subspace topology on \(TU_\alpha \subseteq TS\) agrees with the previous topology on \(TU_\alpha\). i.e., If \(U_\alpha \cap U_\beta \neq \emptyset\), then \((U_\alpha \cap U_\beta) \subseteq TU_\alpha\), and the subspace topology generated by both inclusions agrees.

\(\Rightarrow\) The topology on \(TS\) is 2nd countable and Hausdorff.

Prop: \(TS\) is a smooth manifold with chart \(\{(TU_\alpha, F_\alpha)\}_\alpha\), where
\[F_\alpha := F_{\tau_\alpha(U_\alpha)} \circ G_\alpha: TU_\alpha \rightarrow \mathbb{R}^n\] is smooth.

Pf: N.T.S. \(F_\alpha \circ F_\beta: F_\beta(U_\alpha \cap U_\beta) \rightarrow F_\alpha(U_\alpha \cap U_\beta)\) is smooth.

\[F_\alpha \circ F_\beta (a, b, c, d) = F_{\tau_\alpha(U_\alpha)} \circ G_\alpha \circ G_\beta^{-1} \circ F_{\tau_\beta(U_\beta)}^{-1}\]

Observe: The projection \(\pi: TS \rightarrow S\) is smooth.

\[(p, x) \mapsto p\]

Definition: A smooth vector field on \(S\) is a smooth map \(\sigma: S \rightarrow TS\) s.t. \(\pi \circ \sigma: S \rightarrow S\) is the identity. i.e. \(\sigma(p) = (p, X(p))\) for some \(X(p) \in T_p S\). Sometimes, we denote \(\sigma\) by \(X\).

A Riemannian metric on \(S\) is a family of inner products \(<\cdot, \cdot>_p\) on \(T_p S\) s.t. for smooth vector fields \(X, Y, Z\), the function \(p \mapsto \langle X, Y \rangle_p\) is smooth.
Remark: A smooth vector field can be viewed as a map
\[ X: C^\infty(S) \to C^\infty(S) \]
\[ f \to (p \mapsto X(p)(f)) \]

Given a Riemannian metric \( \langle \cdot, \cdot \rangle_p \) on \( S \), we can talk about length, angle, and area. Let \( \| X \|_p := \sqrt{\langle X, X \rangle_p} \), \( X \in T_p S \).

Definition: Let \( X, Y \in T_p S \). The length of \( X \) is \( \| X \|_p \), and the angle \( \theta \) between \( X \) and \( Y \) is \( \cos^{-1} \left( \frac{\langle X, Y \rangle_p}{\| X \|_p \| Y \|_p} \right) \).

Cauchy-Schwarz inequality: \( \langle X, Y \rangle_p \leq \| X \|_p \| Y \|_p \)

Let \( Y: I \to S \) be a smooth curve. The length of \( Y \),
\[ \lambda(Y) := \int_I \| Y'(t) \|_{Y(t)} \, dt. \]

Let \( U \subseteq \mathbb{R}^2 \) be an open set and \( \Phi: U \to S \) be a diffeomorphism onto its image. The area of \( \Phi \)
\[ A(\Phi) := \int_U \sqrt{1 + \frac{1}{\| x \|_{\Phi(p)}^2} \left( \| x \|_{\Phi(p)}^2 - \frac{1}{2} \langle x, x \rangle_{\Phi(p)} \right)^2} \, dp. \]

where \( \frac{1}{\| x \|_{\Phi(p)}^2} \langle x, x \rangle_{\Phi(p)} = \langle x, x \rangle_{\Phi(p)} \), \( \frac{1}{\| x \|_{\Phi(p)}^2} \langle x, x \rangle_{\Phi(p)} = \langle x, x \rangle_{\Phi(p)} \).

Prop: Let \( Y: I \to S \) and \( \tilde{Y}: I \to S \) be smooth curves with the same image. Then \( \lambda(Y) = \lambda(\tilde{Y}) \).

Pf: Since \( Y'(t) \) and \( \tilde{Y}'(s) \) are non-zero for \( t \in I \), \( s \in \tilde{I} \), \( I \) smooth maps \( f: I \to \tilde{I} \) s.t. \( Y = \tilde{Y} \circ f \)

\[ \lambda(Y) = \int_I \| Y'(t) \|_{Y(t)} \, dt. \]
\[ = \int_I \| (\tilde{Y} \circ f)'(t) \|_{Y(t)} \, dt \]
\[ = \int_I \| \tilde{Y}'(f(t)) \cdot f'(t) \|_{Y(t)} \, dt. \]
\[ = \int_{S} \| q'(s') \|_{q(s')} \cdot |s'(t)| \, dt. \]
\[ = \int_{S} \| q'(s) \|_{q(s)} \, ds. \]

**Exercise:** Show that if \( \varphi_1 : U \rightarrow S \), \( \varphi_2 : U \rightarrow S \) are diffeos onto an open set \( U \subseteq S \), then \( A(\varphi_1) = A(\varphi_2) \).

**Definition:** Let \( \varphi : S \rightarrow \mathbb{R}^3 \) be a smooth immersion, and let \( \langle \cdot, \cdot \rangle \) be the induced Riemannian metric on \( S \), i.e., \( \forall X, Y \in T_pS, \langle X, Y \rangle_p = \langle \varphi_\ast_p(X), \varphi_\ast_p(Y) \rangle_{\mathbb{R}^3} \). The **first fundamental form** of \( \varphi \) at \( p \in S \) is the map \( I_{\varphi(p)} : T_pS \rightarrow \mathbb{R} \)
\[ X \mapsto \|X\|_p^2. \]

**Note:** Can think of the first fundamental form as a map \( I_\varphi : \text{smooth vector fields on } S^2 \rightarrow \text{smooth functions on } S^2 \)
\[ X \mapsto (p \mapsto \|X\|_p^2). \]

Two immersion induce the same Riemannian metric on \( S \) \iff they have the same first fundamental form.

**Example:** Let \( \varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \)
\[ (x_1, x_2) \mapsto (\cos x_1 \sin x_2, x_2) \]
\[ p = (x_0, y_0) \in \mathbb{R}^2. \]

Let \[ \frac{1}{\delta x_1} \bigg|_{\varphi(p)} = \varphi_\ast_{x_1} \left( \frac{1}{\delta x_1} \right), \quad \frac{1}{\delta x_2} \bigg|_{\varphi(p)} = \varphi_\ast_{x_2} \left( \frac{1}{\delta x_2} \right). \]

Then \[ \frac{1}{\delta x_1} \bigg|_{\varphi(p)} = -\sin x_0 \frac{1}{\delta x_1} \bigg|_{\varphi(p)} + \cos x_0 \frac{1}{\delta y_1} \bigg|_{\varphi(p)}. \]

\[ \frac{1}{\delta x_2} \bigg|_{\varphi(p)} = \left( \begin{array}{c} -\sin x_0 \\ \cos x_0 \\ 0 \end{array} \right). \]
\[\left\langle \frac{1}{jz_1}, \frac{1}{jz_1} \right\rangle _p = \left\langle \left( \begin{array}{c} -\sin\theta \\ \cos\theta \end{array} \right), \left( \begin{array}{c} -\sin\theta \\ \cos\theta \end{array} \right) \right\rangle _{e_p} = 1.\]

\[\left\langle \frac{1}{jz_1}, \frac{1}{jz_2} \right\rangle _p = \left\langle \left( \begin{array}{c} 0 \\ i\theta \end{array} \right), \left( \begin{array}{c} 0 \\ i\theta \end{array} \right) \right\rangle _{e_p} = 1.\]

\[\left\langle \frac{1}{jz_1}, \frac{1}{jz_2} \right\rangle _p = 0.\]

\[\left\langle \left( \begin{array}{c} c \\ d \end{array} \right), \left( \begin{array}{c} c \\ d \end{array} \right) \right\rangle _p = ac + bd.\]

\[a \frac{1}{jz_1} + b \frac{1}{jz_2} \Rightarrow I_p \left( \begin{array}{c} a \\ b \end{array} \right) = a^2 + b^2.\]

What is the length of \( C := \{ (x, y, z) \in \Phi(\mathbb{R}^2) : z = 0 \} ? \)

Let \( \gamma : [0, 2\pi] \rightarrow \Phi(\mathbb{R}^2) \)
\[t \mapsto (\cos t, \sin t, 0)\]

Observe that \( \gamma \) is a smooth curve whose image is \( \mathcal{C} \).

\[\lambda(C) = \lambda(\gamma)\]
\[= \int_0^{2\pi} \|\gamma'(t)\|_{\gamma(t)} \, dt\]
\[= \int_0^{2\pi} \|(-\sin t, \cos t)\|_{\gamma(t)} \, dt.\]

\[= 2\pi.\]

What is the area of \( S := \{ (x, y, z) \in \Phi(\mathbb{R}^2) : -1 < z < 1 \} ? \)

Let \( \phi : (-1, 1) \times [0, 2\pi] \rightarrow \Phi(\mathbb{R}^2) \)
\[(s, t) \mapsto (\cos s, \sin t, s)\]

Again, the image of \( \phi \) is \( S \).

\[\text{Area} = \int_{-1}^{1} \int_0^{2\pi} \sqrt{\|(-\sin t, \cos t)\|^2 \| (0) \|^2 - \left\langle \left( \begin{array}{c} \sin t \\ \cos t \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \right\rangle} \, dt \, ds\]
\[= \int_{-1}^{1} \int_0^{2\pi} dt \, ds = 4\pi.\]