Def: i) A \textit{smooth manifold} \((n \geq 2, 3)\) is a second countable, Hausdorff top. space \(M\) together with a maximal collection \(\{(U_a, t_a)\}_a\), where \(t_a: U_a \to \mathbb{R}^n\) are homeomorphisms onto their images \(t_a^{-1}(U_a \cap \mathbb{R}^n)\to t_a(U_a \cap \mathbb{R}^n)\), \(a \in \mathbb{R}^n\), is a diffeomorphism, i.e., a homeo which is smooth and has a smooth inverse.

ii) Each \((U_a, t_a)\) is a chart, and \(\{(U_a, t_a)\}_a\) is a maximal atlas.

iii) A 2-dim smooth manifold is a smooth surface, \(S\).

Note: i) If \(\{(U_a, t_a)\}_a\) is a maximal atlas, then \(\{U_a\}_a\) is an open cover of \(M\).

ii) Let \(\{(U'_a, t'_a)\}_a\) be a collection s.t. \(\{U'_a\}_a\) is an open cover of \(M\) and \(\{t'_a: U'_a \to \mathbb{R}^n\}_a\) satisfy (i). Then \(\exists \) maximal atlas that contains \(\{(U'_a, t'_a)\}_a\).

E.g. i) \(S = \mathbb{R}^2\)

\(\{U'_a\}_a\) any open cover.

\(t'_a = \text{id}: U'_a \to \mathbb{R}^2\).
2) \( S^2 = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 - \frac{1}{2} = 1 \} \).

\( \mathcal{U}_1 = \{ (x, y) \in S^2 : y > -\frac{1}{2} \} \).

\( \mathcal{U}_2 = \{ (x, y) \in S^2 : y < \frac{1}{2} \} \).

\[ \tau_1 : \mathcal{U}_1 \rightarrow \mathbb{R}^2 \]
\[ (x, y) \rightarrow \left( \frac{x}{1 + \frac{1}{2}}, \frac{y}{1 + \frac{1}{2}} \right) \]

\[ \tau_2 : \mathcal{U}_2 \rightarrow \mathbb{R}^2 \]
\[ (x, y) \rightarrow \left( \frac{x}{1 - \frac{1}{2}}, \frac{y}{1 - \frac{1}{2}} \right) \]

\[ \tau_1 \circ \tau_2^{-1} : (x, y) \rightarrow \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right) \]

\[ \tau_2 \circ \tau_1^{-1} : (x, y) \rightarrow \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right) \]

The maximal atlas containing \( \{ (\mathcal{U}_1, \tau_1), (\mathcal{U}_2, \tau_2) \} \) makes \( S^2 \) a smooth surface.

\[ \tau_1 (\mathcal{U}_1 \cap \mathcal{U}_2) = \tau_2 (\mathcal{U}_1 \cap \mathcal{U}_2), \tau_1 \circ \tau_2^{-1} = \tau_2 \circ \tau_1, \text{ an isomorphism about the unit circle.} \]

Exercise: Find an atlas that makes \( \mathbb{T}^2 \) a smooth surface.

\[ ([0,1] \times [0,1]) / \{(0,1) - (1,0) \} \]

\[ (x, y) \rightarrow \left( \frac{x}{k}, \frac{y}{k} \right) \]
Definition: Let $M, N$ be $m$ and $n$ dimensional manifolds. A continuous map $\Psi: M \rightarrow N$ is smooth if it charts $(U_a, \tau_a)$ of $M$ and $(U_b, \tau_b)$ of $N$, the map $t_b' \circ \Psi \circ t_a^{-1}: t_a(U_a \cap \Psi^{-1}(U_b')) \rightarrow t_b'(\Psi(U_a) \cap U_b')$ is smooth.

ii) If $\Psi$ is a bijection and both $\Psi$ and $\Psi^{-1}$ are smooth, then $\Psi$ is a diffeomorphism.

iii) If $f: M \rightarrow \mathbb{R}^n$ is smooth, then it is a smooth function.

Exercise: Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a smooth function. Show that $f|_{S^2}: S^2 \rightarrow \mathbb{R}$ ($S^2$ is equipped with smooth structure defined earlier) is a smooth function.

Exercise: Show that if $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ is a smooth function, then $(x, y, z) \in \mathbb{R}^3, (x, y) \in U, g = f(x, y, z)$ is a smooth surface.

Implicit function theorem

Theorem: Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a smooth function and let $p \in \mathbb{R}^3$. If $\frac{\partial f}{\partial x}(p) \neq 0$, then there exists $\mathbb{R}^3_{\text{open}}$ containing $p$ and a smooth function $g: U \rightarrow \mathbb{R}$ s.t.

$$\{ g \in U : f(g) = f(p) \} = \{ (x, y, z) \in U : g = f(x, y, z) \}.$$ Furthermore, $\frac{\partial g}{\partial z}(p) = \frac{1}{\frac{\partial f}{\partial x}(p)} \forall z \in U.$
Corollary: Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be a smooth function and \( c \in \mathbb{R} \). Then
\[
S := \{ p \in \mathbb{R}^3 : \frac{\partial f}{\partial x}(p) \neq 0 \text{ and } f(p) = c \} \text{ is a (possibly empty) smooth surface.}
\]

Proof: Suppose \( S \neq \emptyset \), then \( \exists p \in \mathbb{R}^3 \) s.t. \( \frac{df}{ds}(p) \neq 0 \) and \( f(p) = c \).

Theorem \Rightarrow \forall p \in S, \exists U_p \subseteq \mathbb{R}^3 \text{ containing } p \text{ and } g_p : \mathbb{R}^2 \to \mathbb{R} \text{ smooth}
\[
\quad \text{s.t. } U_p' = \{ (x,y) \in U_p : g_p(x,y) = f \} = \{ q \in U_p : f(q) = c \}.
\]

\forall U_p, \text{ let } \tilde{f}_p : U_p' \to \mathbb{R}^2 \text{ This is a homeo : } U_p' \text{ is a graph.}
\[
(x,y) \mapsto (x,y).
\]

Notation: \( \{ U_p' \}_{p \in S} \) forms an open cover of \( S \).

Also, if \( U_p' \cap U_q' \neq \emptyset \), then \( \tilde{f}_p \circ \tilde{f}_q^{-1} (x,y) = \tilde{g}_q \circ \tilde{f}_p^{-1} (x,y) \)
\[
= \tilde{g}_q (x,y, g_p(x,y))
= (x,y).
\]

Exercise: Show that \( \{ (x,y,z) \in \mathbb{R}^3 : \sin(x) + e^{x \cos^2 y} = \delta(x,y,z) \} \text{ is a smooth surface.} \)

Corollary: Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be a smooth function and \( c \in \text{Im}(f) \). Then
\[
S := \{ p \in \mathbb{R}^3 : f(p) = c \text{ and } \frac{\partial f}{\partial x}(p) \neq 0 \text{ or } \frac{\partial f}{\partial y}(p) \neq 0 \text{ or } \frac{\partial f}{\partial z}(p) \neq 0 \} \text{ is a smooth surface.}
\]

Example: Let \( f : \mathbb{R}^3 \to \mathbb{R} \)
\[
(x,y,z) \mapsto x^2 + y^2 + z^2.
\]
This $S^2 = f^{-1}(1)$ is a smooth surface.

Why? \( \frac{\partial f}{\partial x} = \sin^2 x \), \( \frac{\partial f}{\partial y} = 2y \), \( \frac{\partial f}{\partial z} = 2z \).

\[ \nabla \neq 0 \implies \nabla f^{-1}(1) \]

\[ \frac{\partial f}{\partial x}(p) \neq 0 \quad \text{or} \quad \frac{\partial f}{\partial y}(p) \neq 0 \quad \text{or} \quad \frac{\partial f}{\partial z}(p) \neq 0. \]

Proof of Cor: Let \( p \in S \).

If \( \frac{\partial f}{\partial x}(p) \neq 0 \), then \( E \) open set \( U_p \subseteq \mathbb{R}^3 \) and smooth function \( f : \mathbb{R}^3 \to \mathbb{R} \)

\[ U_p = \{ (x, y, z) : x = f_p(x, y, z) \} = \{ q \in U_p : f(q) = c \}. \]

Define \( \tau_p : U_p' \to \mathbb{R}^3 \)

\[ (x, y, z) \mapsto (y, z). \]

Similarly, if \( \frac{\partial f}{\partial y}(p) \neq 0 \) (resp. \( \frac{\partial f}{\partial z}(p) \neq 0 \)), \( E \) open \( f : \mathbb{R}^3 \to \mathbb{R} \) and \( U_p' \subseteq \mathbb{R}^3 \)

\[ \tau_p : U_p' \to \mathbb{R}^3 \]

\[ (x, y, z) \mapsto (x, y) \quad (\text{resp. } (x, y, z) \mapsto (x, y)). \]

Clearly, \( \{ U_p' \}_{p \in S} \) is an open cover of \( S \). Let \( p, q \in S \).

\( U_p' \cap U_q' \neq \emptyset \). Assume \( \frac{\partial f}{\partial x}(p) \neq 0 \neq \frac{\partial f}{\partial y}(q) \). Then

\[ \tau_q \circ \tau_p^{-1}(y, z) = \tau_q(f(p), y, z) = (f(q), y, z) \]

This is smooth. \( \square \).

Exercise: Let \( S = \{ (x, y, z) : e^{x^2} = x^2 + y^2 + 1 \} \). Is \( S \) a smooth surface?

**Tangent spaces**

Def: The tangent space to \( \mathbb{R}^n \) at \( p \) is the set of vectors at \( p \). \((x, x, x, \ldots, x)\)
Let \( P : \mathbb{R}^n \to \mathbb{R} \) be a smooth function.

Given a tangent vector \( X \) to \( \mathbb{R}^n \) at \( p \), can differentiate \( f \) in the direction of \( X \) to get \( D_X(f) \).

Explicitly, let \( \frac{\partial}{\partial x_i} \bigg|_p \) be the unit vector at \( p \) parallel to the \( x_i \)-axis. Then
\[
X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \bigg|_p, \quad a_i \in \mathbb{R}, \quad \text{and} \quad D_X(f) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i} (p).
\]

Furthermore,

i) \( D_X(a f + b g) = a D_X(f) + b D_X(g) \) for \( a, b \in \mathbb{R}, \ f, g \in C^\infty(\mathbb{R}^n) \) "linearity."

ii) \( D_X(f \cdot g) = f(p) \cdot D_X(g(p)) + g(p) \cdot D_X(f) \) for \( f, g \in C^\infty(\mathbb{R}^n) \) "product rule."

Definition: A \underline{derivation (in \( \mathbb{R}^n \)) at \( p \)} is a map \( D : C^\infty(\mathbb{R}^n) \to \mathbb{R} \).

(i) and (ii) hold.

\( \therefore \) If \( X \) is a tangent vector, then \( D_X \) is a derivation.

Note: Let \( D_1, D_2 \) be derivations at \( p \), and \( a, b \in \mathbb{R} \). Then the map
\[
(aD_1 + bD_2) : C^\infty(\mathbb{R}^n) \to \mathbb{R}
\]
\[
f \mapsto a D_1(f) + b D_2(f)
\]
is also a derivation.

\( \therefore \) Derivations at \( p \) is a vector space.

Exercise: Show that \( F : \text{ tangent vectors at } p \mathbb{R}^n \to \text{ derivations at } p \mathbb{R} \) is a bijective linear map.

Alternative definition: \( T_p \mathbb{R}^n := \{ \text{derivations at } p \} \), "tangent space to \( \mathbb{R}^n \) at \( p \)."
Definition: Let \( p \in S \). A derivation at \( p \) is a map \( D: C^\infty(S) \rightarrow \mathbb{R} \) s.t. linearity and the product rule hold.

\[ T_pS = \{ \text{derivations at } p \} \], "tangent space to \( S \) at \( p \)."

As before, \( T_pS \) is a vector space.

Example: Let \( (U, \tau) \) be a chart s.t. \( p \in U \) and \( X \in T_{\tau(p)} \mathbb{R}^n \).

\[ D_{x,p}: C^\infty(S) \rightarrow \mathbb{R} \]

is a derivation.

\[ f \mapsto D_{x,p}(f \circ \tau^{-1}) \]

Exercise: Show that \( T_{\tau(p)} \mathbb{R}^n \rightarrow T_pS \) is a linear bijection.

\[ D_x \mapsto D_{x,p} \]

From now on, write \( D_x(f) = X(f) \).

Smooth curves:

**Definition**: Let \( I \subseteq \mathbb{R} \) be an interval. A smooth map \( X: I \rightarrow M \) is a smooth curve if

\[ \frac{d(X(t))}{dt} \bigg|_{t=t_0} \neq 0 \quad \forall t \in I, \text{ and } X \text{ is smooth chart } (\tau_c, U) \text{ of } M. \]
Example: \( \gamma: \mathbb{R} \longrightarrow \mathbb{R}^2 \) is a smooth curve. 
\[ t \longrightarrow (t, t^2) \]

\( \gamma: \mathbb{R} \longrightarrow \mathbb{R}^2 \) is a smooth curve. 
\[ t \longrightarrow (t^2 + t, \sqrt{t} - 1) \]

Def: \( \forall t_0 \in I, \quad \gamma'(t_0) \in T_{\gamma(t_0)} S \) is the derivative \( \gamma'(t_0): C^0(\mathbb{R}) \longrightarrow \mathbb{R} \)
\[ f \longrightarrow \frac{d}{dt} \circ \gamma'(t_0) \]

Example: Let \( \gamma: \mathbb{R} \longrightarrow \mathbb{R}^2 \)
\[ t \longrightarrow (t^2, t^3) = (\gamma_1(t), \gamma_2(t)) \]

\[ \gamma'(t_0) = a \frac{1}{3} \left| \gamma(t_0) \right| + b \frac{1}{6} \left| \gamma(t_0) \right| \]

\[ a = \gamma'(t_0)(x) = \frac{d}{dt} \gamma_1(t_0) = 1 \]

\[ b = \gamma'(t_0)(y) = \frac{d}{dt} \gamma_2(t_0) = 3 t_0 \]

\[ \gamma'(t_0) = \frac{1}{3} \left| \gamma(t_0) \right| + 3 t_0 \frac{1}{6} \left| \gamma(t_0) \right| = \left( \begin{array}{c} 1 \\ 3 t_0 \end{array} \right) = \left( \frac{\gamma_1'(t_0)}{\gamma_2'(t_0)} \right) \]

Let \( f: \mathbb{R}^2 \longrightarrow \mathbb{R} \)
\[ (x, y) \longrightarrow x^2 + y^2 \]

What is \( \gamma'(t_0)(f) ? \)

\[ \gamma'(t_0)(f) = \frac{d}{dt} \circ \gamma'(t_0) \]
\[ = \frac{d}{dt} \left( t^2 + t \right) \bigg|_{t_0} \]
\[ = 2 t_0 + 1 \]

Exercise: Show that \( \forall X \in T_{t_0} S, \exists \varepsilon > 0 \) and a smooth curve \( \gamma: (-\varepsilon, \varepsilon) \longrightarrow S \) s.t. \( \gamma(t_0) = X \).