1. Find an example of an immersion \( \phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) such that there is a point \( p \in \mathbb{R}^2 \) where the principal curvatures have opposite sign.

2. Let \( \phi : S \rightarrow \mathbb{R}^3 \) be a smooth immersion with Gaussian curvature \( K > 0 \) and let \( \gamma : I \rightarrow S \) be a unit speed smooth curve. Then \( ||\gamma''(t)|| \geq \min\{|k_1|, |k_2|\} \) where \( k_1, k_2 \) are principal curvatures at \( \gamma(t) \).

3. Let \( \phi : S \rightarrow \mathbb{R}^3 \) be a smooth immersion and \( \gamma : I \rightarrow S \) be a smooth curve. If there is an (affine) plane in \( \mathbb{R}^3 \) that is tangential to \( \phi(S) \) along \( \phi \circ \gamma(I) \), then \( \gamma(t) \) is parabolic or planar for all \( t \in I \).

4. Let \( \phi : S \rightarrow \mathbb{R}^3 \) be a smooth immersion and let \( p \in S \). Let \( v_1, v_2 \in T_pS \) such that \( ||v_1||_\phi, ||v_2||_\phi = 1 \) and \( \langle v_1, v_2 \rangle_\phi = 0 \). Show that the mean curvature of \( \phi \) at \( p \) is

\[
\frac{1}{2} (\Pi_p(v_1) + \Pi_p(v_2)).
\]

5. Let \( F : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be the map given by \( F : p \mapsto c \cdot p \) for some constant \( c \). Also, let \( \phi : S \rightarrow \mathbb{R}^3 \) be a smooth immersion, let \( \phi_1 := F \circ \phi : S \rightarrow \mathbb{R}^3 \) and let \( q \in S \). Find a formula for the Gaussian curvature of \( \phi_1 \) in terms of the Gaussian curvature of \( \phi \) at the point \( q \).

6. Let \( S_1 \) and \( S_2 \) be smooth surfaces equipped with Riemannian metrics \( g_1, g_2 \) respectively, and let \( \phi : S_1 \rightarrow S_2 \) be a diffeomorphism. Show that \( \phi \) is an isometry if and only if for all smooth curves \( \gamma : I \rightarrow S_1 \), the length of \( \gamma \) is equal to the length of \( \phi \circ \gamma \).

7. Let \( \phi : S \rightarrow \mathbb{R}^3 \) be a smooth immersion and \( \langle \cdot, \cdot \rangle \) the induced Riemannian metric in \( S \). Let \( p \in S \), and let \( (x, y) \) be local coordinates on an open set \( U \) containing \( p \). Define functions \( E, F, G : U \rightarrow \mathbb{R} \) by

\[
E(x, y) := \left\langle \frac{\partial}{\partial x} \bigg|_{(x,y)}, \frac{\partial}{\partial x} \bigg|_{(x,y)} \right\rangle_{(x,y)},
\]

\[
F(x, y) := \left\langle \frac{\partial}{\partial x} \bigg|_{(x,y)}, \frac{\partial}{\partial y} \bigg|_{(x,y)} \right\rangle_{(x,y)},
\]

\[
G(x, y) := \left\langle \frac{\partial}{\partial y} \bigg|_{(x,y)}, \frac{\partial}{\partial y} \bigg|_{(x,y)} \right\rangle_{(x,y)}.
\]

If \( F(x, y) = 0 \) for all \( (x, y) \in U \), show that the Gaussian curvature \( K(p) \) of the immersion \( \phi \) at the point \( p \) is given by the formula

\[
K(p) = -\frac{1}{2\sqrt{E(x, y)G(x, y)}} \left( \frac{\partial}{\sqrt{E(x, y)G(x, y)}} \frac{\partial E(x, y)}{\partial y} + \frac{\partial}{\sqrt{E(x, y)G(x, y)}} \frac{\partial G(x, y)}{\partial x} \right).
\]