MA 108B PROBLEM SET 6 SOLUTIONS

Problem 1 (Wheeden–Zygmund Chapter 5 Problem 4)

As a product of measurable functions, $x^k f(x)$ is measurable and so is $|x^k f(x)|$. Since $0 \leq x^k \leq 1$ on $(0,1)$

$$\int_0^1 |x^k f(x)| \, dx = \int_0^1 x^k |f(x)| \, dx \leq \int_0^1 |f(x)| \, dx < \infty$$

and thus $x^k f(x) \in L(0,1)$.

For the second part, we note that $f_k(x) := x^k f(x)$ can be bounded by $|f_k(x)| \leq |f(x)| \in L(0,1)$. Thus we can apply Lebesgue’s Dominated Convergence theorem to conclude

$$\lim_{k \to \infty} \int_0^1 x^k f(x) \, dx = \lim_{k \to \infty} \int_0^1 f_k(x) \, dx = \int_0^1 \lim_{k \to \infty} f_k(x) \, dx = 0$$

where we used that $f_k(x) \to 0$ a.e. in $(0,1)$ since $x^k \to 0$ on $(0,1)$ and $f$ finite a.e.

Problem 2 (Wheeden–Zygmund Chapter 5 Problem 14)

Since

$$0 \leq \omega_f(\alpha) \leq \omega_f(1)$$

we may assume that $f \geq 0$. The functions

$$f_k(x) := \begin{cases} f(x), & f(x) \leq 1/k \\ 0, & f(x) > 1/k \end{cases}$$

converge to 0 pointwise and thus also $f_k^p \to 0$ ($t \mapsto t^p$ is continuous). Noting that $|f_k^p(x)| \leq |f^p(x)| \in L(E)$ we can apply Lebesgue’s Dominated Convergence theorem to conclude that

$$\int_E f_k^p = \int_{f \leq 1/k} f^p \to 0 \quad (1)$$

as $k \to \infty$. Let $\varepsilon > 0$, then by (1) we can find a $k \in \mathbb{N}$ such that $\int_E f_k^p \leq \varepsilon$. For any $0 < a < 1/k$ we obtain

$$\varepsilon \geq \int_E f_k^p = \int_{f \leq 1/k} f^p \geq \int_{a < f \leq 1/k} f^p \geq a^p \{ a < f \leq 1/k \} = a^p (\omega_f(a) - \omega_f(1/k)).$$

If $a \to 0$ then $a^p \omega_f(1/k) \to 0$. Here we note that $\omega_f$ is finite on $(0, \infty)$ since if $a \in (0, \infty)$ and $\omega_f(a) = \infty$, then $\int_E f^p \geq \int_{f > a} f^p \geq a^p \{ f > a \} = a^p \omega_f(a) = \infty$, contradicting that $f^p \in L(E)$.

$$0 \leq \limsup_{a \to 0} a^p \omega_f(a) \leq \varepsilon.$$  

Since $\varepsilon$ was arbitrary, we can conclude that $\lim_{a \to 0^+} a^p \omega_f(a) = 0$. 

Problem 3 (Wheeden–Zygmund Chapter 5 Problem 15)

Since $\omega_f(\alpha)$ is decreasing we may estimate

$$p \int_{a/2}^{a} \alpha^{p-1} \omega_f(\alpha) \, d\alpha \geq p \omega_f(a) \int_{a/2}^{a} \alpha^{p-1} \, d\alpha = a^p \omega_f(a)(1 - 2^{-p}).$$

Since the above integrands are non-negative the Riemann integrals and Lebesgue integrals are the same, so we get

$$p \int_{a/2}^{a} \alpha^{p-1} \omega_f(\alpha) \, d\alpha \leq p \int_{0}^{a} \alpha^{p-1} \omega_f(\alpha) \, d\alpha \to 0$$

for $a \to 0$ by definition of improper Riemann integrals. If you want to circumvent the notion of improper Riemann integrals, you could also argue that the Lebesgue integral

$$\int_{0}^{a} \alpha^{p-1} \omega_f(\alpha) \, d\alpha \to 0$$

by the monotone convergence theorem. Combining the two results we obtain the desired equality $\lim_{a \to 0} a^p \omega_f(a) = 0$. The case for $b \to \infty$ is similar.

Problem 4 (Wheeden–Zygmund Chapter 5 Problem 16)

We use the same notation as in the proof of Theorem 5.46. We first note that $|E_{ab}| < \infty$ since $\omega_f$ is assumed to be finite. For the restriction of $f$ onto $E_{ab}$ we can construct an increasing sequence of measurable simple functions $f_k \to f|_{E_{ab}}$ as in the proof of Theorem 5.46. Since $f_k^p \to f^p|_{E_{ab}}$ and $|f_k(x)| \leq b$ we can use the bounded convergence theorem ($|E_{ab}| < \infty$) to conclude that

$$\int_{E_{ab}} f_k^p \to \int_{E_{ab}} f^p.$$

Since $f_k^p$ is simple, we compute

$$\int_{E_{ab}} f_k^p = - \sum_j (\alpha_j^{(k)})^p (\omega_{ab}(\alpha_j^{(k)}) - \omega_{ab}(\alpha_{j-1}^{(k)})) = - \sum_j (\alpha_j^{(k)})^p (\omega(\alpha_j^{(k)}) - \omega(\alpha_{j-1}^{(k)}))$$

where we used that $\omega_{ab}(\alpha_j^{(k)}) - \omega_{ab}(\alpha_{j-1}^{(k)}) = \omega(\alpha_j^{(k)}) - \omega(\alpha_{j-1}^{(k)})$ as per the proof of Theorem 5.42. Letting $k \to \infty$ we arrive at

$$\int_{E_{ab}} f^p = - \int_{a}^{b} \alpha^p \, d\omega(\alpha)$$

and subsequently letting $a \to 0$ and $b \to \infty$ we get

$$\int_{E} f^p = - \int_{0}^{\infty} \alpha^p \, d\omega(\alpha)$$

by the monotone convergence theorem. This result is the first part of the statement.

For the second equality, we first use integration by parts

$$\int_{a}^{b} \alpha^p \, d\omega(\alpha) = p \omega(b) - a^p \omega(a) - p \int_{a}^{b} \alpha^{p-1} \omega(\alpha) \, d\alpha.$$

When taking the limit $a \to 0$ and $b \to \infty$, we consider three cases:

If $\int_{0}^{\infty} \alpha^{p-1} \omega(\alpha) \, d\alpha$ is finite, then by Problem 15 $\lim_{b \to \infty} b^p \omega(b) = 0 = \lim_{a \to 0} a^p \omega(a)$, and thus the left-hand-side is finite and the desired equality holds.
If \( \int_0^\infty \alpha^p \, d\omega(\alpha) = \int_E f^p \) is finite, then by Lemma 5.50 and Problem 14 \( \lim_{b \to \infty} b^p \omega(b) = 0 = \lim_{a \to 0} a^p \omega(a) \), and thus the right-hand-side is finite and the desired equality holds.

If neither of the integrals is finite, then the statement holds trivially as \( +\infty = +\infty \).

**Problem 5 (Wheeden–Zygmund Chapter 5 Problem 24)**

(a) We use Tchebyshev’s inequality (5.49) which says that

\[
\omega_{|f|}(\alpha) \leq \alpha^{-p} \int_{|f| > \alpha} |f|^p \leq \alpha^{-p} \int_E |f|^p = \alpha^{-p} A
\]

with \( A = \int_E |f|^p < \infty \). The converse does not necessarily hold. The function \( f(x) = 1/x \) is not integrable on \((0, \infty)\), however the function is in weak \( L^1 \) since

\[
\omega_{|f|}(\alpha) = |\{1/x > \alpha\}| = |\{x < 1/\alpha\}| = 1/\alpha.
\]

(b) Since \( \omega_{|f|} \) is finite, we can use Problem 16 to obtain

\[
\int_E |f|^p = p \int_0^\infty \alpha^{p-1} \omega_{|f|}(\alpha) \, d\omega = p \int_0^1 \alpha^{p-1} \omega_{|f|}(\alpha) \, d\omega + p \int_1^\infty \alpha^{p-1} \omega_{|f|}(\alpha) \, d\omega
\]

\[
\leq A_1 \int_0^1 \alpha^{p-1} \, d\alpha + A_2 \int_1^\infty \alpha^{p-1-r} \, d\alpha
\]

\[
= A_1 \frac{\alpha^{p-1}}{p-1} \bigg|_{\alpha=0}^{\alpha=1} + A_2 \frac{\alpha^{p-r}}{p-r} \bigg|_{\alpha=1}^{\alpha=\infty} < \infty.
\]

(c) Since \( f \) (and consequently also \( |f| \)) is bounded, we know that \( \omega_{|f|}(M) = 0 \) for sufficiently large \( M \). Furthermore \( \omega_{|f|} \) is finite and again we can apply Problem 16 to conclude that

\[
\int_E |f|^p = p \int_0^\infty \alpha^{p-1} \omega_{|f|}(\alpha) \, d\omega = p \int_0^M \alpha^{p-1} \omega_{|f|}(\alpha) \, d\alpha \leq pA \int_0^M \alpha^{p-1-1} \, d\alpha
\]

\[
= pA \frac{\alpha^{p-1}}{p-1} \bigg|_{\alpha=0}^{\alpha=M} < \infty.
\]