MA 108B PROBLEM SET 4 SOLUTIONS

Problem 1 (Wheeden–Zygmund Chapter 3 Problem 6)

Since any open set $G \subset \mathbb{R}^n$ is an element of $\mathcal{B}$, by the properties of a $\sigma$-algebra, we also have that all sets of the form $CG$ are in $\mathcal{B}$. These are precisely all the closed sets. Let $\Sigma$ be a $\sigma$-algebra that contains all the closed sets. Then it contains all the complements of closed sets, i.e. all the open sets. Since $\mathcal{B}$ is the smallest $\sigma$-algebra containing all the open sets, we can conclude that $\mathcal{B} \subset \Sigma$.

Problem 2 (Wheeden–Zygmund Chapter 3 Problem 8)

We first note that all sets involved are measurable if $E_1, E_2$ are. If one of $|E_1|, |E_2|$ is infinite, then the statement trivially holds since $|(E_1 \cup E_2)| \geq \max(|E_1|, |E_2|)$, by the monotonicity of the measure.

Now let $|E_1|, |E_2| < \infty$, then by monotonicity and subadditivity all the involved sets have finite measure. By Corollary 3.25

$$|(E_1 \cup E_2) - E_2| = |E_1 \cup E_2| - |E_2|$$

and

$$|E_1 - (E_1 \cap E_2)| = |E_1| - |E_1 \cap E_2|.$$  

Since $(E_1 \cup E_2) - E_2 = E_1 - (E_1 \cap E_2)$, we can conclude that

$$|E_1 \cup E_2| - |E_2| = |E_1| - |E_1 \cap E_2|$$

and since all terms are finite, we can manipulate the equation to the desired form.

Problem 3 (Wheeden–Zygmund Chapter 3 Problem 11)

Let $E$ be measurable. For $\varepsilon > 0$ we can find an open $G$ such that $E \subset G$ and $|G - E|_e < \varepsilon$. By Theorem 1.11 we can write $G = \bigcup I_k$ with nonoverlapping closed intervals $I_k$. Then

$$|G| = \sum_{k=1}^{\infty} |I_k|.$$  

Since $|E| < \infty$ we have that $|G| = |G - E| + |E| < \infty$ and thus the above series converges. In particular, there exists an $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} |I_k| < \varepsilon$. Define

$$S := \bigcup_{k=1}^{N-1} I_k, \quad N_1 := \bigcup_{k=N}^{\infty} I_k, \quad N_2 := G - E.$$  

Note that $|N_1|_e \leq \sum_{k=N}^{\infty} |I_k| < \varepsilon$ and $|N_2|_e = |G - E| < \varepsilon$. By construction

$$S \cup N_1 - N_2 = G - (G - E) = E$$

where we used that $E \subset G$.  

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Conversely, suppose that for all $\varepsilon > 0$ we can write
\[ E = (S \cap N_1) - N_2 \]
with $S$ a finite union of closed intervals and $|N_1|_e, |N_2|_e < \varepsilon$. For given $\varepsilon > 0$ we aim to construct an open $G$ with $E \subset G$ and $|G - E|_e < \varepsilon$. First choose $S, N_1, N_2$ such that (1) holds with $|N_1|_e, |N_2|_e < \varepsilon/4$. Since $S$ is measurable as a finite union of closed sets, we can find an open $G$ with $S \subset G$ and $|G - S| < \varepsilon/4$. While $N_1$ may not be measurable, we can definitely find an open set $G_1$ such that $N_1 \subset G_1$ and $|G_1| < |N_1|_e + \varepsilon/4 < \varepsilon/2$. Now by (1)
\[ G \cup G_1 - E = G \cup G_1 - ((S \cap N_1) - N_2) \subset (G - S) \cup G_1 \cup N_2 \]
and consequently
\[ |G \cup G_1 - E|_e \leq |(G - S) \cup G_1 \cup N_2|_e \leq |G - S| + |G_1| + |N_2|_e < \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon. \]
Since $G \cup G_1$ is open, and $E \subset G \cup G_1$ we have proved that $E$ is measurable.

**Problem 4 (Wheeden–Zygmund Chapter 3 Problem 13)**

Let $F$ be a closed set such that $F \subset E$. Then by the monotonicity of the outer Lebesgue measure $|F| \leq |E|_e$ (Note that $F$ is measurable since it is closed). Thus
\[ |E|_i = \sup_F |F| \leq |E|_e. \]

Now suppose $|E|_e < \infty$. First, let $E$ be measurable. For $\varepsilon > 0$ we can find a closed set $F$ such that $F \subset E$ and $|E - F| < \varepsilon$ by Theorem 3.22. Then $|E| - |F| = |E - F| < \varepsilon$. Here we used that $E$ is measurable and $|E| < \infty$. Thus $|E| < |F| + \varepsilon$ and we can conclude that $|E|_e = |E| \leq \sup_F |F| + \varepsilon = |E|_i$. We have already seen that the other inequality also holds and thus $|E|_i = |E|_e$.

Conversely, assume that $|E|_i = |E|_e$. For $\varepsilon > 0$ we can find a closed set $F$ and an open set $G$ such that $F \subset E \subset G$ and $|G| < |E|_e + \varepsilon/2$, $|F| > |E|_i - \varepsilon/2$. As a consequence
\[ |G - E|_e \leq |G - F| = |G| - |F| < |E|_e + \varepsilon/2 - |E|_i + \varepsilon/2 = \varepsilon \]
where we used that $|F| < \infty$ and $|E|_i = |E|_e$. This is precisely the definition of measurable.

**Problem 5 (Wheeden–Zygmund Chapter 3 Problem 20)**

Let $E$ be the Vitali set defined in Theorem 3.38. We first note that for any $x \in \mathbb{R}$ there is an $r \in \mathbb{Q}$ such that $x - r \in [0, 1]$. Using the notation of the proof, we can conclude that each equivalence class $E_x$ has a representative in $[0, 1]$. Since $E$ was constructed by choosing exactly one element from each set $E_\alpha$ in a collection $\{E_\alpha \}_{\alpha \in A}$ of distinctive equivalence classes with $\bigcup_{\alpha \in A} E_\alpha = \mathbb{R}$ (here we used Zermelo’s Axiom), the above argument allows us to assume that $E \subset [0, 1]$. The set $\mathbb{Q} \cap [0, 1]$ is countable. Let $\{r_k\}_{k=1}^\infty$ denote an enumeration of these numbers.

Define $E_k = \{x + r_k : x \in E\}$. If $z \in E_k \cap E_j$ then there exists $x, y \in E$ such that $x + r_k = z = y + r_j$. Thus $x - y \in \mathbb{Q}$ and since no two distinct elements of $E$ can be in the same equivalence class, $x = y$, which in turn implies $r_j = r_k$. We conclude that the sets $E_k$ are disjoint. By translation invariance (see last week’s homework: Wheeden–Zygmund Problem 18), we have $|E_k|_e = |E|_e > 0$. Note that $E$ cannot be a set of outer Lebesgue measure zero, as that would imply that $E$ is measurable.
Noting that \( \bigcup_{k \geq 1} E_k \subset [0, 2] \) we obtain

\[
\left| \bigcup_{k \geq 1} E_k \right|_e \leq 2
\]

but

\[
\sum_{k=1}^{\infty} |E_k|_e = \sum_{k=1}^{\infty} |E|_e = +\infty.
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