MA 108B PROBLEM SET 3 SOLUTIONS

Problem 1 (Wheeden–Zygmund Chapter 2 Problem 14)

We consider the pair of functions on $[-1,1]$ given by

$$f(x) = \begin{cases} 0, & -1 \leq x < 0 \\ 1, & 0 \leq x \leq 1 \end{cases}$$

$$\phi(x) = \begin{cases} 0, & -1 \leq x \leq 0 \\ 1, & 0 < x \leq 1 \end{cases}$$

We have already seen in the example on page 34 of the textbook that the Riemann–Stieltjes integral $\int_{-1}^{1} f \, d\phi$ does not exist. For any partition $\Gamma = \{-1 = x_0 < \cdots < x_m = 0\}$ of $[-1,0]$ we compute

$$S_\Gamma = \sum_{i=1}^{m} f(\xi_i) [\phi(x_i) - \phi(x_{i-1})] = 0$$

since $\phi$ is constant on $[-1,0]$. Similarly, for partitions $\Gamma$ of $[0,1]$ we get

$$S_\Gamma = \sum_{i=1}^{m} f(\xi_i) [\phi(x_i) - \phi(x_{i-1})] = f(\xi_1) = 1.$$ 

Since the above sums do not depend on the partition, the Riemann–Stieltjes integral exists.

Problem 2 (Wheeden–Zygmund Chapter 2 Problem 16)

We first consider the two simplest cases of a jump singularity. Let $c \in [a,b]$ and consider the functions

$$f_c^+(x) = \begin{cases} 0, & a \leq x < c \\ 1, & c \leq x \leq b \end{cases} \quad f_c^-(x) = \begin{cases} 0, & a \leq x \leq c \\ 1, & c < x \leq b \end{cases}$$

Now suppose $\phi$ is a function of bounded variation which is continuous at $c$. Let $\Gamma = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be any partition. Then there is an $i$ such that $c \in [x_{i-1}, x_i]$. Then for any selection of intermediate points $\xi_i$ there is a $y \in \{x_{i-1}, x_i, x_{i+1}\}$ such that

$$S_\Gamma(f_c^+, \phi) = \sum_{i=1}^{m} f_c^+(\xi_i) [\phi(x_i) - \phi(x_{i-1})] = \phi(b) - \phi(y).$$

Since $\phi$ is continuous at $c$ and $|c - y| < 2|\Gamma|$ we see that $\int_{a}^{b} f \, d\phi$ exists and equals $\phi(b) - \phi(c)$.

We now consider the general case where $f$ has jump discontinuities at finitely many points $c_1, \ldots, c_k$ (and $\phi$ continuous at each $c_i$). Then we can write $f$ as

$$f = g + \sum_{j=1}^{k} d_j f_{c_j}^+ + \sum_{j=1}^{k} e_j f_{c_j}^-$$

with some coefficients $d_j, e_j$ and a continuous function $g$. To see this, start with $c_1$. If $f(c_1) = f(c_1-) \neq f(c_1+)$ then set $d_1 = 0, e_1 = f(c_1+) - f(c_1)$. Similarly if $f(c_1) = f(c_1+) \neq f(c_1-)$ then set $d_1 = f(c_1) - f(c_1-), e_1 = 0$. Finally if all three numbers are different, then set
$d_1 = f(c_1) - f(c_1^-), e_1 = f(c_1^+) - f(c_1)$. The function $f - d_1 f_{c_1}^+ - e_1 f_{c_1}^-$ only has jump discontinuities at $c_2, \ldots, c_k$. Proceed iteratively.

From the simple case above and Theorem 2.16 we conclude that

$$\lim_{|Γ|→0} S_Γ\left(\sum_j d_j f_{c_j}^+ + \sum_j e_j f_{c_j}^-, φ\right)$$

exists, and by Theorem 2.24 so does $\lim_{|Γ|→0} S_Γ(g, φ)$. Evoking again Theorem 2.16 we can conclude that $\int_D f \, dφ$ exists.

**Problem 3 (Wheeden–Zygmund Chapter 3 Problem 3)**

With $D_0 = [0, 1] × [0, 1]$ let $D_k$ be the set of points remaining after $k$ steps of the process and set $D = \bigcap_k D_k$. The set $D_k$ is a union of $4^k$ closed squares, each of area $9^{-k}$, i.e. $D_k = \bigcup_{j=1}^{4^k} Q_j^k$. Thus $D_k$ is closed and as an intersection of closed sets, $D$ is closed. By Theorem 3.18 (or Theorem 3.14) it is measurable. Since $|D| ≤ |D_k| ≤ 4^k 9^{-k}$ for all $k ≥ 1$, it holds that $|D| = 0$.

Let $a_j^k, b_j^k, c_j^k, d_j^k$ be the corners of $Q_j^k$. Then each of these points will be the corner of some cube of $D_{k+1}$ and thus $a_j^k, b_j^k, c_j^k, d_j^k \in D$. In particular $D$ is not empty.

Since $D$ is closed it remains to show that every point of $D$ is a limit point of $D$ in order for $D$ to be perfect (and thus uncountable). Let $x \in D$, then for any $k ≥ 1$ by the definition of $D$, there exists a cube $Q_{jk}^k$ with $x \in Q_{jk}^k$. To construct a sequence of points $x_k$ in $D$ that converges to $x$, choose for example

$$x_k = \begin{cases} a_{jk}^k, & x \neq a_{jk}^k \\ b_{jk}^k, & x = a_{jk}^k. \end{cases}$$

As seen above $x_k \in D$ and $|x - x_k| ≤ \text{diam}(Q_{jk}^k) = 3^{-k} \sqrt{2}$. Thus $x_k → x$.

To show that $D = C × C$ we note that $D_k = C_k × C_k$. This is easy to see and can be proved formally by induction: For $k = 1$, the statement clearly is true. Now let $x = (x_1, x_2) \in D_{k+1}$. Then $x$ is in one of the four corner subsquares of some cube $x \in Q_{j0}^k$ of $D_k$. The projection of $Q_{j0}^k$ onto the $x$-axis lies by the induction hypotheses in $C_k$. Since $x$ is in one of the corner subsquares, $x_1$ is in either the first third or the last third of the projection onto the $x$-axis and thus by definition in $C_{k+1}$. Similarly $x_2 \in C_{k+1}$ and arguing backwards we also see that $x = (x_1, x_2) \in D_{k+1}$ if $x_1 \in C_{k+1}$ and $x_2 \in C_{k+1}$. Finally,

$$D = \bigcap_k D_k = \bigcap_k (C_k × C_k) = \left(\bigcap_k C_k\right) × \left(\bigcap_k C_k\right) = C × C.$$

**Problem 4 (Wheeden–Zygmund Chapter 3 Problem 4)**

Fix $θ$. With $D_0 = [0, 1]$ let $D_k$ be the set of points remaining after $k$ steps of the process and set $D = \bigcap_k D_k$. The set $D_k$ is a union of $2^k$ closed intervals, each of length $(1-θ)^k$, i.e. $D_k = \bigcup_{j=1}^{2^k} I_j^k$. Thus $D_k$ is closed and as an intersection of closed sets, $D$ is closed. By Theorem 3.18 (or Theorem 3.14) it is measurable. Since $|D| ≤ |D_k| ≤ 2^k (1-θ)^k 2^{-k} = (1-θ)^k$ for all $k ≥ 1$, it holds that $|D| = 0$ (Note that $\lim_{k→∞} (1-θ)^k = 0$).

Let $a_j^k, b_j^k$ be the endpoints of $I_j^k$. Then each of these points will be the endpoint of some interval of $D_{k+1}$ and thus $a_j^k, b_j^k \in D$. In particular $D$ is not empty.
Since $D$ is closed it remains to show that every point of $D$ is a limit point of $D$ in order for $D$ to be perfect (and thus uncountable). Let $x \in D$, then for any $k \geq 1$ by the definition of $D$, there exists an interval $I^k_{jk}$ with $x \in I^k_{jk}$. To construct a sequence of points $x_k$ in $D$ that converges to $x$, choose for example
\[
x_k = \begin{cases} a^k_{jk}, & x \neq a^k_{jk} \\ b^k_{jk}, & x = a^k_{jk}. \end{cases}
\]
As seen above $x_k \in D$ and $|x - x_k| \leq \text{diam}(I^k_j) = (1 - \theta)2^{-k}$. Thus $x_k \to x$.

**Problem 5 (Wheeden–Zygmund Chapter 3 Problem 18)**

First note that for any $n$-dimensional interval $I = \{x : a_i \leq x_i \leq b_i \mid i = 1, \ldots, n\}$, clearly
\[
|I|_e = v(I) = \prod_{i=1}^{n} (b_i - a_i) = \prod_{i=1}^{n} (b_i + h - a_i - h) = v(I_h) = |I_h|_e.
\]

Let $S = \{I_k\}$ be a countable cover of $E$ consisting of intervals. Then $S' = \{(I_k)_h\}$ is a countable cover of $E_h$ consisting of intervals and $\sigma(S) = \sum_k v(I_k) = \sum_k v((I_k)_h) = \sigma(S')$. Thus for any cover $S$ of $E$ there is a cover $S'$ of $E_h$ with $\sigma(S) = \sigma(S')$ and by the definition of the outer Lebesgue measure we conclude $|E|_e \geq |E_h|_e$. Noting that $E = (E_h)_h$ and repeating the argument yields $|E_h|_e \geq |E|_e$ and thus $|E|_e = |E_h|_e$.

If $E$ is measurable then for $\varepsilon > 0$ there exists an open $G$ with $E \subset G$ and $|G - E|_e < \varepsilon$. The set $G_h$ is open (for a formal argument note that $G_h = f^{-1}(G)$ for the continuous function $f(x) = x - h$) and $E_h \subset G_h$. Furthermore $G_h - E_h = (G - E)_h$ and thus $|G_h - E_h|_e = |G - E|_e < \varepsilon$, so $E_h$ is measurable.