MA 108B PROBLEM SET 2 SOLUTIONS

Problem 1 (Wheeden–Zygmund Chapter 2 Problem 4)

Suppose we have a sequence of functions \( f_k : [a, b] \to \mathbb{R} \) which converges to \( f \) pointwise, with \( V[f_k, a, b] \leq M \) for all \( k \). Then let \( \varepsilon > 0 \) be given, and let \( \Gamma = \{a = x_0 < x_1 < \cdots < x_m = b\} \) be a any partition. Due to the pointwise convergence, we can choose \( k \) so large that
\[
\sum_{i=0}^{m} |f_k(x_i) - f(x_i)| < \varepsilon/2.
\]

Then we can calculate
\[
S_{\Gamma}(f) = \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| \\
= \sum_{i=1}^{m} |f(x_i) - f_k(x_i) + f_k(x_i) - f_k(x_{i-1}) + f_k(x_{i-1}) - f(x_{i-1})| \\
\leq \sum_{i=1}^{m} (|f(x_i) - f_k(x_i)| + |f_k(x_i) - f_k(x_{i-1})| + |f_k(x_{i-1}) - f(x_{i-1})|) \\
= S_{\Gamma}(f_k) + \sum_{i=1}^{m} |f_k(x_i) - f(x_{i-1})| + \sum_{i=1}^{m} |f_k(x_{i-1}) - f_k(x_{i-1})| \\
\leq M + \varepsilon
\]

Since \( \Gamma \) was arbitrary, this shows \( V[f, a, b] \leq M + \varepsilon \) and since \( \varepsilon > 0 \) was arbitrary, we conclude \( V[f, a, b] \leq M \).

For an example of a bounded sequence of functions with bounded variation which converges to a function with unbounded variation, recall the function \( f(x) = x \sin(1/x) \) from Problem 5 last week (Wheeden–Zygmund Chapter 2 Problem 1) and define the sequence of functions
\[
f_k(x) = \begin{cases} 
  x \sin(1/x), & x \geq 1/k \\
  0, & x < 1/k.
\end{cases}
\]

Clearly \( f_k \to f \) pointwise. To see that \( f_k \) is of bounded variation, note that by Theorem 2.2
\[
V[f_k, 0, 1] = V[f_k, 0, 1/k] + V[f_k, 1/k, 1] = |f_k(1/k)| + V[f_k, 1/k, 1]
\]

Since \( f_k \) is continuously differentiable on \([1/k, 1]\), Corollary 2.10 allows us to conclude that \( f_k \) is indeed of bounded variation on \([0, 1]\). We have already seen, however, that \( f \) is of unbounded variation.
Problem 2 (Wheeden–Zygmund Chapter 2 Problem 5)

Consider any partition \( \Gamma = \{a = x_0 < x_1 < \cdots < x_m = b\} \) of \([a, b]\) and let \(C = |f(a)| + |f(b)| + M\). Then \(\Gamma' = \{x_1 < x_2 < \cdots < x_m = b\}\) is a partition of \([x_1, b]\), so by the assumption with \(\varepsilon = x_1 - a\) we get

\[
S_\Gamma(f) = |f(x_1) - f(x_0)| + \sum_{i=2}^m |f(x_i) - f(x_{i-1})| \\
= |f(x_1) - f(x_0)| + S_{\Gamma'}(f) \\
\leq C + M
\]

where, we used that

\[
|f(x_1) - f(x_0)| \leq |f(x_1) - f(b)| + |f(b)| + |f(a)| \leq C.
\]

Thus \(V[f, a, b] < \infty\).

We do not, in general, have that \(V[f, a, b] \leq M\). For instance, consider the function

\[
f(x) = \begin{cases} 0, & 0 < a \leq b \\ 1, & x = a. \end{cases}
\]

Clearly \(f\) restricted to \([a + \varepsilon, b]\) has variation 0, but \(f\) itself has variation 1. If, however \(f\) is continuous at \(a\), we get this stronger result. To this end, we note that if \(\Gamma'\) is a refinement of \(\Gamma\) (i.e. \(\Gamma \subset \Gamma'\)), then \(S_\Gamma(f) \leq S_{\Gamma'}(f)\) by the triangle inequality. Since \(f\) is continuous at \(a\) there is \(\delta > 0\) such that \(|f(c) - f(a)| < \varepsilon\) if \(|c - a| < \delta\). For any partition \(\Gamma = \{a = x_0 < x_1 < \cdots < x_m = b\}\) consider the refinement \(\Gamma' = \Gamma \cap \{a + \delta/2\} = \{a = x_0' < x_1' < \cdots < x_{m+1}' = b\}\).

Then definitely \(|x_1' - x_0'| = |x_1' - 1| \leq |a + \delta/2 - a| = \delta/2\) and thus

\[
S_{\Gamma'}(f) = S_{\Gamma}(f) = |f(x_1') - f(x_0')| + \sum_{i=2}^{m+1} |f(x_i') - f(x_{i-1}')| \leq \varepsilon + M.
\]

Since \(\varepsilon > 0\) was arbitrary, this shows \(V[f, a, b] \leq M\).

Problem 3 (Wheeden–Zygmund Chapter 2 Problem 6)

On \([0, \varepsilon]\) the function \(f(x) = x^2 \sin(1/x)\) is continuously differentiable and thus by Corollary 2.10

\[
V[f, \varepsilon, 1] = \int_\varepsilon^1 |f'(x)| \, dx = \int_\varepsilon^1 |2x \sin(1/x) - \cos(1/x)| \leq \int_\varepsilon^1 (2x + 1) \, dx \leq 2.
\]

The result of problem 2 (Wheeden–Zygmund Chapter 2 Problem 5), allows us to conclude that \(V[f, 0, 1] < \infty\).

Problem 4 (Wheeden–Zygmund Chapter 2 Problem 13)

For

\[
R_\Gamma(f, \phi) = \sum_{j=1}^m f(x_j)(\phi(x_j) - \phi(x_{j-1})
\]
it is straightforward to show that
\[ R_\Gamma(cf, \phi) = cR_\Gamma(f, \phi), \]
\[ R_\Gamma(f, c\phi) = cR_\Gamma(f, \phi), \]
\[ R_\Gamma(f_1 + f_2, \phi) = R_\Gamma(f_1, \phi) + R_\Gamma(f_2, \phi), \]
\[ R_\Gamma(f, \phi_1 + \phi_2) = R_\Gamma(f, \phi_1) + R_\Gamma(f, \phi_2). \]

To finish the proof:

- If \( c = 0 \) the statement is trivial. Otherwise, for \( \varepsilon > 0 \) choose \( \delta > 0 \) such that 
  \[ |R_\Gamma(f, \phi) - \int_a^b f \, d\phi| < \varepsilon/|c| \text{ for } |\Gamma| < \delta. \] 
  Then for \( |\Gamma| < \delta \),
  \[ |R_\Gamma(cf, \phi) - c\int_a^b f \, d\phi| = |c| |R_\Gamma(cf, \phi) - \int_a^b f \, d\phi| < \varepsilon. \]

An analogous argument works for \( \int f \, d(c\phi) \).

- For \( \varepsilon > 0 \) choose \( \delta > 0 \) such that \( |R_\Gamma(f_1, \phi) - \int_a^b f_1 \, d\phi| < \varepsilon/2, |R_\Gamma(f_2, \phi) - \int_a^b f_2 \, d\phi| < \varepsilon/2 \) for \( |\Gamma| < \delta \). Then for \( |\Gamma| < \delta \)
  \[ |R_\Gamma(f_1 + f_2, \phi) - \int_a^b f_1 \, d\phi - \int_a^b f_2 \, d\phi| \]
  \[ \leq |R_\Gamma(f_1, \phi) - \int_a^b f_1 \, d\phi| + |R_\Gamma(f_2, \phi) - \int_a^b f_2 \, d\phi| < \varepsilon. \]

An analogous argument works for \( \int f \, d(\phi_1 + \phi_2) \).

PROBLEM 5 (Wheeden–Zygmund Chapter 2 Problem 27)

If \( V[f, -1, 1] = +\infty \), then by Theorem 2.6 \( P[f, -1, 1] = +\infty \) and the statement trivially holds. If \( V[f, -1, 1] < \infty \) then by the same theorem
\[ P[f, -1, 1] = \frac{1}{2}(V[f, -1, 1] + f(b) - f(a)) = \frac{1}{2}V[f, -1, 1] \]
since \( f(x) = f(-x) \).

By the above and Theorem 2.2
\[ 2P[f, -1, 1] = V[f, -1, 1] = V[f, -1, 0] + V[f, 0, 1]. \quad (1) \]

Assume that \( \Gamma = \{0 = x_0 < \cdots < x_m = 1\} \) is a partition of \([0, 1]\), then \( \Gamma' = \{-1 = -x_m < \cdots < -x_0 = 0\} \) is a partition of \([-1, 0]\) and we compute
\[ V_\Gamma = \sum_{j=1}^m |f(x_k) - f(x_{k-1})| = \sum_{j=1}^m |f(-x_{k-1}) - f(-x_k)| = V_{\Gamma'} \leq V[f, -1, 0]. \]

This implies \( V[f, -1, 0] \geq V[f, 0, 1] \) and repeating the above argument backwards starting with the partition \( \Gamma' \), we also get \( V[f, -1, 0] \leq V[f, 0, 1] \). Together with (1) we conclude that \( P[f, -1, 1] = V[f, 0, 1] \).
Lastly assume that $\Gamma = \{0 = x_0 < \cdots < x_m = 1\}$ is a partition of $[0, 1]$, then $\Gamma' = \{-1 = -x_m < \cdots < -x_0 = 0\}$ is a partition of $[-1, 0]$ and since $x^+ = (-x)^-$ we compute

$$P_\Gamma = \sum_{j=1}^{m} (f(x_k) - f(x_{k-1}))^+ = \sum_{j=1}^{m} (f(-x_k) - f(-x_{k-1}))^+
$$

$$= \sum_{j=1}^{m} (f(-x_{k-1}) - f(-x_k))^- = N_{\Gamma'} \leq N[f, -1, 0]$$

This implies $N[f, -1, 0] \geq P[f, 0, 1]$ and repeating the above argument backwards starting with the partition $\Gamma'$, we also get $N[f, -1, 0] \leq P[f, 0, 1]$. 