MIDTERM EXAMINATION FOR MA 108B WINTER 2016/17  
(DUE THURSDAY FEBRUARY 9TH 2017 4:00PM)  

Instructor: Lukas Schimmer, Sloan 380

INSTRUCTIONS

Please read the following carefully before starting the exam.

• Due to federal privacy policies, the exam is to be submitted with a cover sheet, which carries nothing other than your full name.
• Write your full name on each sheet of paper you use for the exam. Enumerate the pages, to make clear which parts belong together. Staple the exam before turning it in.
• The exam is to be completed within 4 hours in a single sitting.
• The exam must be turned in by 4:00pm February 9th 2017, in the homework drop box next to the Math Office.
• The exam is open-book so you may use your notes for 108b, the textbook and the solutions posted on the course webpage.
• No discussion about the exam with anyone (other than me, if something is unclear) is permitted.
• Clearly explain your line of argument, including notation you use (if different from class) and objects you introduce. Give a sufficient amount of details that makes your line of reasoning self-contained and clear; if in doubt, it is advisable to rather include more than too little details.
• All five problems are equally counted.

Good luck!
Problem 1

Suppose \( f, \phi_1, \phi_2 \) are continuous and of bounded variation on \([a, b] \). Show that

\[
\int_a^b f \, d(\phi_1 \phi_2) = \int_a^b f \phi_1 \, d\phi_2 + \int_a^b f \phi_2 \, d\phi_1.
\]

Problem 2

Let \( BV[a, b] \) denote the set of all finite, real-valued functions of bounded variation on \([a, b] \). From the results in the textbook we know that \( BV[a, b] \) is a vector space.

(a) Prove that \( \|f\|_{BV} := V[f; a, b] + |f(a)| \) defines a norm on \( BV[a, b] \), i.e. for all \( f, g \in BV[a, b] \) and \( \lambda \in \mathbb{R} \):

- \( 0 \leq \|f\|_{BV} < \infty \)
- \( \|f\|_{BV} = 0 \) if and only if \( f = 0 \)
- \( \|\lambda f\|_{BV} = |\lambda| \|f\|_{BV} \)
- \( \|f + g\|_{BV} \leq \|f\|_{BV} + \|g\|_{BV} \)

(b) Prove that \( BV[a, b] \) is complete with respect to this norm, i.e. if \( f_k \in BV[a, b] \) is a Cauchy sequence (\( \forall \varepsilon > 0 \exists N \in \mathbb{N} \) such that \( \|f_k - f_m\|_{BV} < \varepsilon \) if \( k, m \geq N \)) then there is a function \( f \in BV[a, b] \) with \( \|f_k - f\|_{BV} \to 0 \) as \( k \to \infty \).

Hint: You may use that any uniformly Cauchy sequence of bounded, real-valued functions on \([a, b] \) converges uniformly to a bounded, real-valued function.

Problem 3

Let \( C \) be a curve with parametric equations \( x = \phi(t), y = \psi(t) \) and \( a \leq t \leq b \). If \( \phi \) and \( \psi \) are of bounded variation and continuous on \([a, b] \), show that the length of \( C \) is given by

\[
L(C) = \lim_{|\Gamma| \to 0} l(\Gamma).
\]

Problem 4

(a) Let \( \{E_k\}_{k=1}^m \) be a finite collection of disjoint measurable subsets of \( \mathbb{R}^n \) and let \( S \) be any subset of \( \mathbb{R}^n \) (not necessarily measurable). Show that

\[
\left| S \cap \bigcup_{k=1}^m E_k \right|_e = \sum_{k=1}^m |S \cap E_k|_e.
\]
(b) Let \( \{E_k\}_{k=1}^{\infty} \) be a countable collection of disjoint measurable subsets of \( \mathbb{R}^n \) and let \( S \) be any subset of \( \mathbb{R}^n \) (not necessarily measurable). Show that

\[
\left| \left| S \cap \bigcup_{k=1}^{\infty} E_k \right| \right|_e = \sum_{k=1}^{\infty} |S \cap E_k|_e .
\]

**Problem 5**

Find an example of a measurable subset \( E \) of \([0, 1]\) such that \( |E| = 0 \), yet its difference set \( \{d : d = x - y, x \in E, y \in E\} \) contains a non-empty, open interval centered at the origin.

*Hint: Copy the construction of the Cantor set \( C \), but in the cube \( Q = [0, 1] \times [0, 1] \). First, remove all but four closed cubes of side length 1/3, one at each corner of \( Q \); then, repeat this procedure in each of the remaining cubes (see Figure 1) and so on. It was proved in the homework that the resulting set is perfect, has Lebesgue measure 0 and is precisely \( C \times C \). For a fixed \( \alpha \in [-1, 1] \), consider the line \( y = x + \alpha \) in the plane. Prove that the line intersects the constructed set.*