6b: HW3 Solutions

The first write-up is adapted after Mary Giambrone.

1. Recall that $G$ is 2-connected means that $G$ can be constructed repeatedly by adding $G$-paths to a cycle. It is important to think about this characterization as an if and only if condition, since we will argue that at least one of $H_1$ or $H_2$ is 2-connected by checking that this structure is preserved in at least one of them. If the edge $e$ is in the cycle but a $G$-path does not end on both of its vertices, then contracting that edge will not affect the connectivity of the graph, i.e. $H_2$ will still consist of a cycle plus some $G$-paths; in particular, this means that $H_2$ is still 2-connected. Likewise, if $e$ is part of one of the $G$-paths, and the endpoints are both not also the endpoints in another $G$-path, then contradicting it is also acceptable, because the $G$-path $e$ originally lied on will still be a $G$-path, connected to some cycle and to other $G$-paths.

If $e$ is an edge in the cycle or a $G$-path where two of the endpoints of a $G$-path are at the vertices defining $e$, then $H_2$ might not be 2-connected anymore (because now the $G$-path is connected to the cycle via the same vertex; hence the removal of this sole vertex might disconnect $H_2$). Nonetheless, the graph $G$ will remain 2-connected if you remove $e$, because this operation will just form a larger cycle or $G$-path, so $H_1$ still consists of a cycle and $G$-paths, which makes it 2-connected. □

The next two write-ups are adapted after Tatiana Brailovskaya.

2. Considering $C$ to be a largest cycle in $G$. For sake of contradiction, assume that $|C| < 2k$; otherwise we are done. Since $G$ is $k$-connected, we know that the minimum degree has to be $k$. Thus, the degree of each vertex in $V$ has to be at least $k$. From this information, we know that the size of the largest cycle in $G$ has to be at least $k + 1$. Let’s show why this holds. Start at vertex $v_1$. Pick any of its $k$ (or more) neighbors. Repeat the process until you reach some vertex $v_i$ that has all of its neighbors in the cycle (including $v_1$). If $i \leq k$, then it has at most $k - 1$ other members of the cycle as its neighbors (including $v_1$). We know that $v_i$ has at least degree $k$, therefore it is a neighbor of at least one vertex $v_{i+1}$ that is not yet in the cycle. Thus, $i > k$. Thus, the length of the longest cycle in $G$ is $k + 1$.

Now, consider $C$ and some vertex $x \in V \setminus C$. We will show that there are at least $k$ vertex-distinct paths between $x$ and $C$ that also terminate at different vertices. To show this, consider $G' = (V', E')$ s.t. $V' = V \cup v$ where vertex $v$ is connected to each of the vertices in $C$ by an edge. Let’s show that $G'$ has to be $k$-connected. Suppose that it is not. Then, there exists a set of $k - 1$ vertices that disconnects $G'$. One of those vertices has to be $v$, since removing $k - 1$ vertices from $V$ will not disconnect $G$, a subgraph of $G'$ and will not disconnect $v$ from $G$ since $k$ neighbors of $v$ have to removed in order for $v$ to become disconnected from $G$. The order in which we remove vertices does not affect whether $G'$ becomes disconnected or not. Start by removing $v$. Next, we are left with a $k$-connected subgraph $G$ of $G'$. If we remove $k - 2$ vertices from $G$ it will not become disconnected. Thus, $G'$ has to be $k$-connected.

Finally, let’s show that there are at least $k$ vertex-distinct paths between $x$ and $C$ that also terminate at different vertices. First, there have to be at least $k$ vertex distinct paths between $x$ and $v$ by Menger’s theorem because $x$ and $v$ cannot be adjacent since $x \notin C$, therefore the theorem applies. Since the paths are vertex disjoint, each path crosses $k$ different vertices right before it reaches $v$. Each of those vertices is in $C$. If we remove $v$, those paths remain in place, since they traverse only edges/vertices that are in $G$ up until the final edge that takes each path to $v$. Thus, there are at least $k$ vertex-distinct paths between $x$ and $C$ that also terminate at different vertices.
Finally, note that since there are $k$ paths from $x$ to $C$ having only vertex $x$ in common, by pigeonhole principle there is at least one pair of adjacent vertices in $C$ joined to $x$ by two of these paths (since $|C| < 2k$). If we join these paths together and remove the edge between these two neighbors, we extend cycle $C$ by at least one vertex $x$, possibly more if the paths consist of more than one edge. This is however impossible, since $C$ was chosen to be the largest cycle in the graph. We have reached a contradiction.

We have thus proven that we can find a cycle of length at least $2k$ in a $k$-connected graph with at least $2k$ vertices.

3. First, let’s compute the number of edges in such $G$. Since there is an equal number of vertices of deg 3 and 4 and those are the only vertices $\in V$, we get

$$|E| = \frac{1}{2} \left( 3 \frac{|V|}{2} + 4 \frac{|V|}{2} \right) = \frac{7}{4}|V|.$$ 

Thus, $|V|$ has to be divisible by 4 for us to get an integer number of edges in $G$. Additionally, $K_5$ consists of 5 vertices of degree 4. Since vertex suppression and edge/vertex deletion cannot increase degree of a vertex, $G$ must contain at least 5 vertices of degree 4 to begin with. Since there is an equal amount of vertices of degrees 4 and 3, there also have to be at least 5 vertices of degree 3 in $G$. Thus, the smallest possible number of vertices of $G$ is 10. The smallest number that is $\geq 10$ and divisible by 4 is 12. Thus, 12 is the smallest possible size of $G$. It is not difficult to find an example of a graph on 12 vertices satisfying the requirements.

4. The statement is false, since duals of 2-connected plane graphs with minimum degree 3 may still have parallel edges. For instance, consider a square with vertices 1, 2, 3, 4 (the sides of the rectangle are edges), and then add four new vertices outside of the square: $x, y$ next to the side 12 and $z, t$ next to the side 34. Also, add new edges \{(x, 1), (x, 2), (y, 1), (y, 2), (y, x), (z, 3), (z, 4), (t, 3), (t, 4), (z, t)\}. The dual of the graph thus obtained has two parallel edges coming from the faces incident to sides 23 and 41 of the square.

5. (a) For sake of contradiction, let $G = (V, E)$ be a 2-connected simple plane graph with at least 3 faces whose dual $G^* = (V^*, E^*)$ is not 2-connected. This means that there is a vertex $v^* \in V^*$ whose removal allows $V^*$ to be partitioned into two disjoint sets $S^*$ and $T^*$ such that there are no edges in $E^*$ between $S^*$ and $T^*$. In $G$, this corresponds to two disjoint sets of faces $F_S$ and $F_T$ such that no face in $S$ shares an edge with a face in $T$. In order for this to happen, $f$, the face in $G$ that corresponds to $v^*$ must fully enclose a nonempty set of faces, which must be either $F_S$ or $F_T$. Suppose, without loss of generality, that this set of faces is $F_S$. Let $S$ and $T$ represent the vertices on the boundaries of the faces $F_S$ and $F_T$, respectively. Then, note that $S$ and $T$ can share at most one vertex, otherwise $f$ is not a complete face. If they do share one vertex, removing that vertex disconnects $G$. If they do not share any vertices, there can be at most one edge from $S$ to $T$, in which case removing either endpoint of that edge disconnects $G$. Either way, we get a contradiction.

(b) The statement is false. And an example is provided by (almost) any 4-connected (simple) plane graph with at least five faces that you can draw. For instance, take a triangle together with its midpoints and its circumcircle. The 6 vertices in the picture are the 6 vertices of your graph, and the edges are given by all small segments and the 3 arcs of the circumcircle.