Non-simple Graphs

- In this class we allow graphs to be non-simple.
- We allow parallel edges, but not loops.
Incidence Matrix

- Consider a graph $G = (V, E)$.
  - We order the vertices as $V = \{v_1, v_2, ..., v_n\}$ and the edges as $E = \{e_1, e_2, ..., e_m\}$.
  - The **incidence matrix** $M_{ij}$ of $G$ is an $n \times m$ matrix $M$. The cell $M_{ij}$ contains 1 if $v_i$ is an endpoint of $e_j$, and 0 otherwise.

\[
M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
\]

Playing with an Incidence Matrix

- Let $M$ be the incidence matrix of a graph $G = (V, E)$, and let $B = MM^T$.
  - What is the value of $B_{ii}$? The **degree** of $v_i$ in $G$.
  - What is the value of $B_{ij}$ for $i \neq j$? The **number** of edges between $v_i$ and $v_j$.

\[
M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
\]
Adjacency Matrix

- Consider a graph $G = (V, E)$.
  - We order the vertices as $V = \{v_1, v_2, ..., v_n\}$.
  - The **adjacency matrix** of $G$ is a symmetric $n \times n$ matrix $A$. The cell $A_{ij}$ contains the number of edges between $v_i$ and $v_j$.

$$A = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 3 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 3 & 0 & 1 & 0
\end{pmatrix}$$

A Connection

- **Problem.** Let $G = (V, E)$ be a graph with incidence matrix $M$ and adjacency matrix $A$. **Express** $MM^T$ **using** $A$.
- **Answer.** $MM^T$ is a $|V| \times |V|$ matrix.
  - $(MM^T)_{ii}$ is the degree of $v_i$.
  - $(MM^T)_{ij}$ for $i \neq j$ is number of edges between $v_i$ and $v_j$.
  - Let $D$ be a diagonal matrix with $D_{ii}$ being the degree of $v_i$. We have $MM^T = A + D$. 
Example

\[
A = \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 3 \\
1 & 3 & 0 \\
\end{pmatrix}
\]

\[
M = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

\[
D = \begin{pmatrix}
2 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 4 \\
\end{pmatrix}
\]

\[
M M^T = \begin{pmatrix}
2 & 1 & 1 \\
1 & 4 & 3 \\
1 & 3 & 4 \\
\end{pmatrix} = A + D
\]

Multiplying by 1’s

- Let \(1_n\) denote the \(n \times n\) matrix with 1 in each of its cells.

- **Problem.** Let \(G = (V, E)\) be a graph with adjacency matrix \(A\). Describe the values in the cells of \(B = A1_{|V|}\).

- **Answer.** It is a \(|V| \times |V|\) matrix.
  - The column vectors of \(B\) are identical.
  - The \(i\)'th element of each column is the degree of \(v_i\).
Playing with an Adjacency Matrix

Let $A$ be the adjacency matrix of a simple graph $G = (V, E)$, and let $B = A^2$.

- What is the value of $B_{ii}$? The degree of $v_i$ in $G$.
- What is the value of $B_{ij}$ for $i \neq j$? The number of vertices that are adjacent to both $v_i$ and $v_j$.

$$A = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0
\end{pmatrix}$$

What about $A^3$?

Let $A$ be the adjacency matrix of a simple graph $G = (V, E)$. For $i \neq j$:

- $A_{ij}$ tells us if there is an edge $(v_i, v_j) \in E$.
- $(A^2)_{ij}$ tells us how many vertices are adjacent to both $v_i$ and $v_j$.
- What is $(A^3)_{ij}$? It is the number of paths of length three between $v_i$ and $v_j$.
- In fact, $(A^2)_{ij}$ is the number of paths of length two between $v_i$ and $v_j$. 
The Meaning of $A^k$

- **Theorem.** Let $A$ be the adjacency matrix of a graph $G = (V, E)$. Then $(A^k)_{ij}$ is the number of paths of length $k$ between $v_i$ and $v_j$.

An Example

![Graph Diagram]

$A = \begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix}$

$A^2 = AA = \begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix}\begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix} = \begin{pmatrix}
2 & 0 & 2 & 0 \\
0 & 2 & 0 & 2 \\
2 & 0 & 2 & 0 \\
0 & 2 & 0 & 2
\end{pmatrix}$

$A^3 = A^2A = \begin{pmatrix}
2 & 0 & 2 & 0 \\
0 & 2 & 0 & 2 \\
2 & 0 & 2 & 0 \\
0 & 2 & 0 & 2
\end{pmatrix}\begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 4 & 0 & 4 \\
4 & 0 & 4 & 0 \\
0 & 4 & 0 & 4 \\
4 & 0 & 4 & 0
\end{pmatrix}$
The Meaning of $A^k$

- **Theorem.** Let $A$ be the adjacency matrix of a graph $G = (V, E)$. Then $(A^k)_{ij}$ is the number of paths of length $k$ between $v_i$ and $v_j$.

- **Proof.** By induction on $k$.
  - **Induction basis.** Easy to see for $k = 1$ and $k = 2$.

The Induction Step

- We have $A^k = A^{k-1}A$. That is
  \[(A^k)_{ij} = \sum_{m=1}^{\lvert V \rvert} (A^{k-1})_{im} A_{mj}\]
  - **By the induction hypothesis,** $(A^{k-1})_{im}$ is the number of paths of length $k - 1$ between $v_i$ and $v_m$.
  - $A_{mj}$ is the number of edges between $v_m, v_j$. 
The Induction Step (cont.)

\[(A^k)_{ij} = \sum_{m=1}^{\vert V \vert} (A^{k-1})_{im}A_{mj}\]

- For a fixed \(m\), \((A^{k-1})_{im}A_{mj}\) is the number of paths from \(v_i\) to \(v_j\) of length \(k\) with \(v_m\) as their penultimate vertex.
  - Summing over every \(1 \leq m \leq \vert V \vert\) results in the number of paths from \(v_i\) to \(v_j\) of length \(k\).

Which British Classic Rock Band is More Sciency?

[Images of LED-ZEPPELIN, QUEEN, and THE BEATLES]
Computing the Number of Paths of Length $k$

- **Problem.** Consider a graph $G = (V, E)$, two vertices $s, t \in V$, and an integer $k > 0$. Describe an algorithm for **finding the number of paths of length $k$ between $s$ and $t$**.
  - Let $A$ be the adjacency matrix of $G$.
  - We need to compute $A^k$, which involves $k - 1$ matrix multiplication.

Matrix Multiplication: A Brief History

- We wish to multiply **two $n \times n$ matrices**.
  - Computing one cell requires about $n$ multiplications and additions. So computing an entire matrix takes $cn^3$ (for some constant $c$).
  - In 1969, **Strassen** found an improved algorithm with a running time of $cn^{2.807}$.
  - In 1987, **Coppersmith and Winograd** obtained an improved $cn^{2.376}$.
  - After over 20 years, in 2010, **Stothers** obtained $cn^{2.374}$.
  - **Williams** immediately improved to $cn^{2.3728642}$.
  - In 2014, **Le Gall** improved to $cn^{2.3728639}$. 
A More Efficient Algorithm

- To compute $A^k$, we do not need $k - 1$ matrix multiplications.

- If $k$ is a power of 2:
  - $A^2 = AA, A^4 = A^2A^2, \ldots, A^k = A^{k/2}A^{k/2}$.
  - Only $\log_2 k$ multiplications!

- If $k$ is not a power of 2:
  - We again compute $A, A^2, A^4, \ldots, A^{|k|}$.
  - We can then obtain $A^k$ by multiplying a subset of those.
  - For example, $A^{57} = A^{32}A^{16}A^{8}A$.
  - At most $2\log_2 k - 1$ multiplications!

Connectivity and Matrices

- **Problem.** Let $G = (V, E)$ be a graph with an adjacency matrix $A$. Use $B = I + A + A^2 + A^3 + \cdots + A^{|V| - 1}$ to tell whether $G$ is connected.

- **Answer.**
  - $G$ is connected iff every cell of $B$ is positive.
  - The main diagonal of $B$ is positive due to $I$.
  - A cell $B_{ij}$ for $i \neq j$ contains the number of paths between $v_i$ and $v_j$ of length at most $|V| - 1$. If the graph is connected, such paths exist between every two vertices.
And Now with Colors

**Problem.** Consider a graph $G = (V, E)$, two vertices $s, t \in V$, and an integer $k > 0$. Moreover, every edge is colored either red or blue. Describe an algorithm for finding the number of paths of length $k$ between $s$ and $t$ that have an even number of blue edges.

**Solution**

- We define two sets of matrices:
  - Cell $ij$ of the matrix $E^{(m)}$ contains the number of paths of length $m$ between $v_i$ and $v_j$ using an even number of blue edges.
  - Cell $ij$ of the matrix $O^{(m)}$ contains the number of paths of length $m$ between $v_i$ and $v_j$ using an odd number of blue edges.

- **What are $E^{(1)}$ and $O^{(1)}$?**
  - $E^{(1)}$ is the adjacency matrix of $G$ after removing the blue edges. Similarly for $O^{(1)}$ after removing the red edges.
Solution (cont.)

- We wish to compute \( E^{(k)} \).
  - We know how to find \( E^{(1)} \) and \( O^{(1)} \).
  - How can we compute \( E^{(m)} \) using \( E^{(m-1)} \) and \( O^{(m-1)} \)?
    \[ E^{(m)} = E^{(m-1)}E^{(1)} + O^{(m-1)}O^{(1)}. \]
  - \( (E^{(m-1)}E^{(1)})_{ij} \) is the number of paths of length \( m \) between \( v_i \) and \( v_j \) with an even number of blue edges and whose last edge is red.
  - Similarly for \( (O^{(m-1)}O^{(1)})_{ij} \) except that the last edges of the paths is blue.

Solution (cont.)

- How can we similarly compute \( O^{(m)} \) using \( E^{(m-1)} \) and \( O^{(m-1)} \)?
  \[ O^{(m)} = E^{(m-1)}O^{(1)} + O^{(m-1)}E^{(1)}. \]

- **Concluding the solution.**
  - We compute about \( 2k \) matrices.
  - Each computation involves two matrix multiplications and one addition.
Identical Graphs?

- Are the following two graphs identical?

\[ \begin{array}{c}
\bullet & v_3 & \bullet \\
\bullet & v_1 & \bullet \\
\bullet & v_2 & \\
\end{array} \quad \begin{array}{c}
\bullet & v_3 & \bullet \\
\bullet & v_1 & \bullet \\
\bullet & v_2 & \\
\end{array} \]

- Possible answers:
  - No, since in one \( v_1 \) has degree 2 and in the other degree 1.
  - Yes, because they have exactly the same structure.

Graph Isomorphism

- An isomorphism from a graph \( G = (V, E) \) to a graph \( G' = (V', E') \) is a bijection \( f: V \to V' \) such that for every \( u, v \in V \), \( E \) has the same number of edges between \( u \) and \( v \) as \( E' \) has between \( f(u) \) and \( f(v) \).
  - We say that \( G \) and \( G' \) are isomorphic if there is an isomorphism from \( G \) to \( G' \).
  - In our example, the graphs are isomorphic \((f(v_1) = u_2, f(v_2) = u_1, f(v_3) = u_3)\).
Uniqueness?

- **Question.** If two graphs are isomorphic, can there be **more than one isomorphism** from one to the other?
  - Yes!
  - \(a \rightarrow w, b \rightarrow z, c \rightarrow x, d \rightarrow y.\)
  - \(a \rightarrow y, b \rightarrow x, c \rightarrow z, d \rightarrow w.\)

Isomorphisms and Adjacency Matrices

- **What can we say about adjacency matrices of isomorphic graphs?**
  
  \[
  \begin{pmatrix}
  0 & 1 & 0 & 0 \\
  1 & 0 & 1 & 0 \\
  0 & 1 & 0 & 1 \\
  0 & 0 & 1 & 0 \\
  \end{pmatrix}
  \quad \begin{pmatrix}
  0 & 0 & 0 & 1 \\
  0 & 0 & 1 & 1 \\
  0 & 1 & 0 & 0 \\
  1 & 1 & 0 & 0 \\
  \end{pmatrix}
  \]
Answer

- Two graphs $G, G'$ are isomorphic iff the adjacency matrix of $G$ is obtained by permuting the rows and columns of the adjacency matrix of $G'$.
  - (The same permutation should apply both to the rows and to the column).
  - $1 \rightarrow 1, 2 \rightarrow 4, 4 \rightarrow 3, 3 \rightarrow 2$:

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\quad \rightarrow \quad
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
\end{pmatrix}
$$

The End: Queen

- Freddie Mercury
  - Degree in Biology
  - Electronics engineer
  - Ph.D. in astrophysics