Crossing Numbers

- The **crossing number** \( cr(G) \) of a graph \( G = (V, E) \) is the minimum number of **pairs of crossing edges** that a planar drawing of \( G \) can have.
- What graphs have a crossing number of zero? **Planar graphs**.
- What is \( cr(K_5) \)? 1.
The Crossing Number of $K_6$

- **Problem.** Find $cr(K_6)$.
- **Solution.**
  - Recall that any planar graph with $n$ vertices has at most $3n - 6$ edges.
  - That is, any planar subgraph of $K_6$ has at most 12 edges.

Solution (cont.)

- We can draw at most 12 edges of $K_6$ without a crossing.
  - $K_6$ has 15 edges, and adding each of the three remaining edges yields at least one crossing.
  - So $cr(K_6) \geq 3$.
  - The following figure shows that $cr(K_6) = 3$. 

[Diagram of $K_6$ graph with 6 vertices and 12 edges drawn without crossing]
A First Bound

- **Claim.** For any graph $G = (V, E)$, we have $cr(G) \geq |E| - (3|V| - 6)$.
- **Proof.**
  - Consider a drawing of $G$ that minimizes the number of crossings.
  - We first draw a maximum plane subgraph that is contained in this drawing. This subgraph has at most $3|V| - 6$ edges.
  - Adding every additional edge increases the number of crossings by at least one. There are at least $|E| - (3|V| - 6)$ such edges.

Is This a Good Bound?

- For simple graphs, the bound $cr(G) \geq |E| - 3|V| + 6$ is smaller than $|V|^2 / 2$.
  - Is this close to the maximum number of crossings that is possible?
  - To find the maximum possible number of crossings, we consider $cr(K_n)$. The above bound implies

$$cr(K_n) \geq \frac{n(n - 1)}{2} - 3n + 6 \approx \frac{n^2}{2}.$$
Estimating $cr(K_n)$

- **Theorem.** For sufficiently large $n$, we have
  \[
  \frac{n^4}{120} - cn^3 \leq cr(K_n) < \frac{n^4}{24},
  \]
  for some constant $c$.

Upper bound

- **Trivial!**
  - Every crossing is the intersection of two edges, which are defined by four vertices.
  - The number of ways to choose four vertices is
    \[
    \binom{n}{4} < \frac{n^4}{24}.
    \]
Lower Bound

- Consider a drawing of $K_n$ that minimizes the number of crossings.
  - Removing any one vertex results in a drawing of $K_{n-1}$. This drawing has at least $cr(K_{n-1})$ crossings.
  - We have $n$ different drawings of $K_{n-1}$, and together they contain at least $n \cdot cr(K_{n-1})$ crossings.
  - Each crossing is counted exactly $n - 4$ times. Thus, we have
    \[(n - 4) \cdot cr(K_n) \geq n \cdot cr(K_{n-1}).\]

Lower Bound (cont.)

- We prove by induction on $n$ that $cr(K_n) \geq \frac{1}{5} \binom{n}{4}$.
  - **Induction basis.** For $n = 5$, we know that $cr(K_5) = 1 = \frac{1}{5} \binom{5}{4}$.
  - **Induction step.** By the previous slide
    \[
    cr(K_n) \geq \frac{n}{n-4} \cdot cr(K_{n-1}) \geq \frac{n}{n-4} \cdot \frac{1}{5} \binom{n-1}{4}
    \]
    \[
    = \frac{n \cdot 1}{n-4} \cdot \frac{(n-1)(n-2)(n-3)}{5 \cdot 4!}
    \]
    \[
    = \frac{n}{n-4} \cdot \frac{1}{5} \binom{n}{4}
    \]
The Correct Bound

- Somewhat more involved arguments lead to $cr(K_n) \approx \frac{n^4}{64}$.
- It is conjectured that
  
  $$cr(K_{m,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor,$$

  but the problem remains open.

  ◦ This is known as the brick factory problem, since it was asked by Turán while doing forced labor in a brick factory during World War II.

So How Bad is Our Bound?

- We have the bound
  
  $$cr(G) \geq |E| - 3|V| + 6.$$

- This implies that $cr(K_n) \geq \frac{n^2}{2}$.
- We have $cr(K_n) \approx \frac{n^4}{64}$.
- For large $n$ the bound is significantly smaller than the actual value.
An Improved Bound

**Theorem.** Let $G = (V, E)$ be a graph with $|E| \geq 4|V|$. Then

$$cr(G) \geq \frac{|E|^3}{64|V|^2}.$$ 

We consider a drawing of $G$ with a minimum number of crossings $c$. Set $p = \frac{4|V|}{|E|}$.

- $S \subset V$ – the subset obtained by independently choosing each vertex of $V$ with probability $p$.
- $c_S$ – the number of crossings that remain in the drawing of $G$ after removing $V \setminus S$.
- $G_S = (S, E_S)$ – the subgraph induced on $S$.
- $\mathbb{E}[c_S] = p^4 c$.

By linearity of expectation

$$\mathbb{E}[c_S - |E_S| + 3|S|] = p^4 c - p^2|E| + 3p|V|$$

$$= \frac{4^4|V|^4 c}{|E|^4} - \frac{16|V|^2}{|E|} + \frac{12|V|^2}{|E|}.$$
\(G_S = (S, E_S)\) – the subgraph induced on \(S\).

\(c_S\) – the number of crossings that remain in the drawing of \(G\) after removing \(V \setminus S\).

\[
\mathbb{E}[c_S - |E_S| + 3|S|] = \frac{4^4|V|^4c - 4|V|^2}{|E|^4} - \frac{4|V|^2}{|E|}.
\]

Thus, there exists a set \(S \subset V\) with

\[
c_S - |E_S| + 3|S| \leq \frac{4^4|V|^4c - 4|V|^2}{|E|^4} - \frac{4|V|^2}{|E|}.
\]

**By the weak bound**, \(c_S \geq |E_S| - 3|S| + 6\).

Thus, \(\frac{4^4|V|^4c}{|E|^4} \geq \frac{4|V|^2}{|E|} + 6 > \frac{4|V|^2}{|E|}\). That is,

\[
c > \frac{|E|^3}{64|V|^2}.
\]

**A Minor Detail**

Where in the proof did we use the restriction \(|E| \geq 4|V|\)?

- The probability for choosing a vertex is \(p = \frac{4|V|}{|E|}\). When \(|E| < 4|V|\), this is not well defined.
Is This Bound Better?

- We have $cr(G) \geq \frac{|E|^3}{64|V|^2}$.
  - That is $cr(K_n) \geq \frac{(\frac{n^2}{2})^3}{64n^2} = \frac{n^8}{2^5}$.
  - Recall that $cr(K_n) \approx \frac{n^4}{64}$.
  - Even though there is a gap in the constants, the dependency on $n$ is correct.

Point-Line Incidences

- $L$ – a set of lines.
- $P$ – a set of points.
- An incidence: $(p, \ell) \in P \times L$ so that $p \in \ell$. 

15 incidences
Lower Bound

- Erdős. By taking a $\sqrt{m} \times \sqrt{m}$ integer lattice and the $n$ lines that contain the largest number of points, we have $c(m^{2/3}n^{2/3} + m + n)$ incidences.

The Szemerédi–Trotter Theorem

- Theorem. The number of incidences between any set $P$ of $m$ points and any set $L$ of $n$ lines is at most $c(m^{2/3}n^{2/3} + m + n)$. 
$I$ – the number of incidences.

We build a graph.

° A vertex for every point.
° An edge between two vertices if they are consecutive on a line.

A line that is incident to $k$ points yields $k - 1$ edges. Thus, the number of edges is $I - n$.

° By the crossing lemma, the number of crossings in the graph is at least $\frac{(I-n)^3}{64m^2}$.

° Since every two lines intersect at most once, the number of crossings is $< \frac{n^2}{2}$.

$$\frac{(I-n)^3}{64m^2} < \frac{n^2}{2} \quad \rightarrow \quad I < 32^{1/3}m^{2/3}n^{2/3} + n$$

A Minor Issue

° The lower bound on the number of crossings applies only when $|E| \geq 4|V|$.

° Since $|E| = I - n$, if $|E| < 4|V|$ then $I < n + 4m$. 
Shameless Advertising!

- In the spring quarter, Adam will teach a class about incidences.
  - These problems involve very interesting mathematics.
  - Check it out!

The Unit Distances Problem

- Problem (Erdős ’46). How many pairs of points in a set of $n$ points could be at unit distance from each other?
  - By taking $n$ points evenly spaced on a line, we have $n - 1$ unit distances.
Early Results

- **Erdős** showed that a $\sqrt{n} \times \sqrt{n}$ square lattice with the right choice of distances determines $n^{1+c/\log \log n}$ unit distances, for some constant $c$.

- Erdős also proved that any set of $n$ points determines at most $cn^{3/2}$ unit distances.

An Improved Result

- Although in the past 70 years MANY top combinatorists worked on the problem, only one work managed to improve the bound (**Spencer, Szemerédi, and Trotter 1984**).

  - **Theorem.** Every set of $n$ points determines at most $cn^{4/3}$ unit distances.
Incidences with Circles

- Given a set of circles and a set of points, an incidence is a pair \((p, C)\) where \(p\) is a point, \(C\) is a circle, and \(p\) is contained in \(C\).

- 11 incidences are in the figure.

Unit Distances and Unit Circles

- We place a unit circle around every point.
- The number of point-circle incidences is twice the number of unit distances.
- Thus, it suffices to find an upper bound for the number of incidences between \(n\) points and any \(n\) unit circles.
Incidence Bound

- **Theorem.** There are at most $cn^{4/3}$ incidences between any set $P$ of $n$ points and any set $C$ of $n$ unit circles.

Building a Graph

- We build a graph.
  - A vertex for every point.
  - An edge between two points if they are consecutive along at least one circle.
  - A circle that is incident to $k$ points yields at least $k - 1$ edges.
  - An edge can originate from at most two circles.
Double Counting Crossings

- $l$ – the number of point-circle incidences.
- We have a graph with $n$ vertices and at least $(l - n)/2$ edges.
- The number of crossings in the graph at least \[
\frac{|E|^3}{64|V|^2} \geq \frac{(l - n)^3}{2^9 n^2}.
\]
- Since any two circles intersect at most twice, the number of crossings is smaller than $n^2$.
- Combining the two implies $n^2 \geq \frac{(l - n)^3}{2^9 n^2}$. That is $(l - n)^3 \leq 2^9 n^4$ or $l \leq 2^3 n^{4/3} + n$.

The End