Kuratowski’s Theorem

**Theorem.** A graph is planar if and only if it does not have $K_5$ and $K_{3,3}$ as topological minors.

- We know that if a graph contains $K_5$ or $K_{3,3}$ as a topological minor, then it is **not planar**.
- It remains to prove that every non-planar graph contains such a topological minor.

Kazimierz Kuratowski
**Reminder: Topological Minors**

- A graph $H$ is a *topological minor* of a graph $G$ if $G$ contains a subdivision of $H$ as a subgraph.

![Diagram of topological minor](image1)

**Reminder: Kuratowski Subgraphs**

- Given a graph $G$, a *Kuratowski subgraph* of $G$ is a subgraph that is a subdivision of $K_5$ or $K_{3,3}$.

![Diagram of Kuratowski subgraph](image2)
3-Connectedness

- **Lemma.** Let $G = (V, E)$ be a graph with fewest edges among all non-planar graphs without a Kuratowski subgraph. Then $G$ is 3-connected.
  - We proved this lemma in the previous class.
  - To complete the proof of Kuratowski’s Theorem, we prove that every 3-connected graph without a Kuratowski subgraph is planar.

Already Proved

- **Lemma A.** Let $G = (V, E)$ be a 3-connected graph with $|V| \geq 5$. Then there exists an edge $e \in E$ whose contraction results in a 3-connected graph.
- **Lemma B.** Let $G = (V, E)$ be a graph with no Kuratowski subgraph. Then contracting any edge $e \in E$ gives a graph with no Kuratowski subgraph.
Recall: Convex Polygons

• A polygon is *convex* if no line segment between two of its vertices intersects the outside of the polygon.
• Equivalently, every interior angle of a convex polygon is smaller than $\pi$.

Convex Polygons

Convex

Not Convex

Convex Embeddings

• A *convex embedding* of a planar graph $G$ is a plane graph of $G$ with all the edges being straight-line segments and in which the boundary of every face is convex.

  ◦ We now investigate which planar graphs have a convex embedding.
Convex Embeddings and Connectivity

- It is known that the planar graph $K_{2,4}$ has no convex embedding. The connectivity of $K_{2,4}$ is 2.
  - On the other hand, for 3-connected graphs this cannot be the case.

3-connected Graph and Convex Embeddings

- **Theorem (Tutte).** If $G = (V, E)$ is 3-connected and contains no Kuratowski subgraph, then $G$ has a convex embedding with no three vertices on a line.
  - This implies that $G$ is planar, and thus completes the proof of Kuratowski’s theorem.
Proof

- We prove the claim by **induction on** \(|V|\).
  - **Induction basis.** The smallest 3-connected graph has four vertices.
  - The only 3-connected graph with four vertices:

![Diagram of a 4-vertex graph]

Induction Step

- **By Lemma A**, since \(G\) is 3-connected, there exists an edge \(e\) whose contraction results in a 3-connected graph \(G_e\).
  - **By Lemma B**, \(G_e\) also has no Kuratowski subgraph.
  - **By the induction hypothesis**, there is a convex embedding of \(G_e\) with no three points on a line.
Induction Step (cont.)

- Let \( z \) be the vertex of \( G_e \) obtained by contracting \( e = (x, y) \).
  - Since \( G_e \) is 3-connected, removing \( z \) results in a face \( f \) with a boundary that is a polygon.
  - Since \( G_e - z \) is 2-connected, the boundary of \( f \) is connected.
  - If a vertex \( v \) of this polygon is connected to \( z \) in \( G_e \), then \( v \) is connected to \( x \) or to \( y \) (or to both) in \( G \).

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A Convex Embedding of \( G \)

- Let \( x_1, \ldots, x_k \) be the neighbors of \( x \), arranged in a cyclic order according to their position in \( f \).
- If every neighbor of \( y \) is between \( x_i \) and \( x_{i+1} \) (possibly including \( x_i \) and \( x_{i+1} \)), we can expand \( e \) back and obtain a convex embedding of \( G \).
Two Remaining Cases

- Two cases remain:
  - $x$ and $y$ have at least three common neighbors.
  - There are vertices $x_i, y_j, x_k, y_\ell$ in this order around the boundary.
- The first case cannot happen since it would imply that $G$ contains a subdivision of $K_5$.

The Last Case

- It remains to consider the case where there vertices $x_i, y_j, x_k, y_\ell$ are in this order around the boundary.
  - This case also cannot happen, since then $G$ would contain a subdivision of $K_{3,3}$.

$V_1 = \{x, y_1, y_2\}$
$V_2 = \{y, x_1, x_2\}$
Conclusion

• The only case that did not lead to a contradiction is the first one, in which we have a convex embedding of $G$.
  ◦ That is, there always exists a convex embedding of $G$.

Fáry’s Theorem

• Theorem. Any simple planar graph can be drawn without crossings so that its edges are straight line segments.
  ◦ We do not prove the theorem in this course.
Reminder: Map Coloring

• Can we color each face with one of four colors, so that no two adjacent faces have the same color?

Reminder: The Four Color Theorem

• **Theorem.** Every map has a 4-coloring.
  ◦ Asked over 150 years ago.
  ◦ Over the decades several false proofs were published.
  ◦ Proved in 1976 by **Appel and Haken.** Extremely complicated proof that relies on a computer program.
Map Coloring and Graphs

- Place a **vertex** in each **face**.
- Place an **edge** between any pair of vertices that represent **adjacent faces**.
  - **The problem.** Can we color the vertices using four colors, such that every edge is adjacent to two different colors?

Maps and Planar Graphs

- **A graph that is obtained from a map is planar** (we can easily draw non-crossing edges).
- Similarly, we can reduce the problem of coloring a planar graph $G$ to a problem of coloring the faces of a map.
  - Coloring the **vertices of $G$** is equivalent to coloring the **faces of the dual graph** $G^*$. 
Coloring Planar Graphs

- Since coloring planar graphs and coloring maps are equivalent problems, the four color theorem states that every planar graph can be colored with four colors.
  - This proof is too complicated for us (or for any living person...).
  - Instead, we prove that every planar graph can be colored by using five colors.

Warm Up Problem

- **Problem.** Prove that any planar graph $G = (V, E)$ can be colored by using six colors.

- **Hint.** Recall a planar graph has at most $3|V| - 6$ edges.
Proof by Induction

- We prove the claim by induction on $|V|$.
  - Since $|E| \leq 3|V| - 6$, the average degree is $\frac{2|E|}{|V|} \leq 6 - \frac{12}{|V|}$.
  - Thus, there is a vertex $v \in V$ of degree smaller than six in $G$.
  - We remove $v$ to obtain the planar graph $G_1 = (V_1, E_1)$. By the hypothesis, $G_1$ can be colored using six colors.
  - We place $v$ back. Since the degree of $v$ is at most five, there is a valid color for it.

Five Colors

- **Problem.** Prove that any planar graph $G = (V, E)$ can be colored by using five colors.
Proof by Induction

- We start in the same way: Proof by induction on $|V|$.
  - There is a vertex $v \in V$ of degree at most five in $G$.
  - We remove $v$ to obtain the planar graph $G_1 = (V_1, E_1)$. By the hypothesis, $G_1$ can be colored using five colors.
  - We place $v$ back. If it is of degree at most four, we can color it. **What if $v$ is of degree five?**

The Case of Degree 5

- Consider the case where $v$ is of degree 5 and each of its neighbors is of a different color.
  - If removing $v$ disconnects $G$, we can permute the colors in one of the components.
The Problematic Case

- A neighbor $a$ of $v$ is colored with color 1, and we would like to change it to color 2.
  - Let $b$ be a neighbor of $v$ with color 2.
  - A problem occurs if there is a path between $a$ and $b$ that is alternately colored using colors 1 and 2.

The Problematic Case (cont.)

- It two neighbors of $v$ do not have an alternating chain between them, we can give both the same color, and then color $v$.
  - It is impossible to have both an $a$-$c$ alternating chain and a $b$-$d$ alternating chain without the two crossing.
  - Thus we can always color $v$, which completes the proof.
The End

DAMN IT, WILLIAMS. STOP PLAYING VIDEO GAMES AT WORK!

 Gimme a sec. I just have to fight this Boss.

Alright, I'm back.