Recall: Plane Graphs

- A plane graph is a drawing of a graph in the plane such that the edges are non-crossing curves.
Recall: Planar Graphs

- The drawing on the left is not a plane graph. However, on the right we have a different drawing of the same graph, which is a plane graph.
- An abstract graph that can be drawn as a plane graph is called a planar graph.

Non-Planar Graphs

- Recall. We proved that \( K_5 \) and \( K_{3,3} \) are not planar.
  - Thus, every graph that contains \( K_5 \) or \( K_{3,3} \) as a subgraph is also not planar.
  - Are there graphs that do not contain \( K_5 \) and \( K_{3,3} \) as subgraphs, and are not planar?
  - Yes, and we can use \( K_5 \) and \( K_{3,3} \) to generate them.
More Non-Planar Graphs

- **Subdividing edges** of $K_5$ or $K_{3,3}$ cannot make them planar.
  - If we have a plane drawing after the subdivision, the same drawing works for the original graph.

Reminder: Topological Minors

- A graph $H$ is a **topological minor** of a graph $G$ if $G$ contains a subdivision of $H$ as a subgraph.
Kuratowski's Theorem

Theorem. A graph is planar if and only if it does not have $K_5$ and $K_{3,3}$ as topological minors.

- We know that if a graph contains $K_5$ or $K_{3,3}$ as a topological minor, then it is not planar.
- It remains to prove that every non-planar graph contains such a topological minor.

Kazimierz Kuratowski

Minimal Non-planar Graph

- A minimal non-planar graph is a non-planar graph $G$ such that any proper subgraph of $G$ is planar.
- What minimal non-planar graphs can you think of?
  - $K_5$ and $K_{3,3}$. 
**Kuratowski Subgraphs**

- Given a graph $G$, a *Kuratowski subgraph* of $G$ is a subgraph that is a subdivision of $K_5$ or of $K_{3,3}$.

**Proof Strategy**

- To prove *Kuratowski's theorem*, we need to prove that every non-planar graph contains a Kuratowski subgraph.
  - It suffices to prove this only for *minimal non-planar graphs*.
- **Strategy:**
  - Show that every minimal non-planar graph with no Kuratowski subgraph *must be 3-connected*.
  - Then show that every 3-connected graph with no Kuratowski subgraph *is planar*.
Choosing the Unbounded Face

**Lemma.** Let $G$ be a planar graph, and let $F$ be a set of edges that form the boundary of a face in an embedding of $G$. Then there exists a non-crossing drawing of $G$ where $F$ is the boundary of the unbounded face.

**Proof**

- We draw the graph on a sphere, and then project it from a point on the face $f$.
  - In the projection on the plane, $f$ will be the unbounded face.
Bad Math Joke #1

- **Q:** What do you call a young eigensheep?
- **A:** A lamb, duh!

Connectedness of Minimal Non-planar Graphs

- **Claim.** Every minimal non-planar graph is 1-connected.
  - Assume for contradiction that there exists a minimal non-planar graph $G$ that is not connected.
  - Let $C$ be a connected component of $G$.
  - By the minimality of $G$, both $C$ and $G - C$ are planar.
  - But then we can draw $C$ and then draw $G - C$ inside one of its faces. **Contradiction!**
2-Connectedness

- **Claim.** Every minimal non-planar graph is **2-connected**.
  - Assume for contradiction that there exists a minimal non-planar graph $G = (V, E)$ that is not 2-connected.
  - There exists a vertex $v$ whose removal disconnects $G$.
  - Let $C$ be a component of $G - v$.
  - By the minimality of $G$, the induced subgraph on $C \cup \{v\}$ and $(V \setminus C) \cup \{v\}$ are both planar.
  - We can embed both graphs with $v$ on the unbounded face, and merge both copies of $v$.

**Illustration**

![Illustration](image-url)
Preparing for 3-Connectedness

- **Claim.** Let $G \in (V, E)$ be a non-planar graph and let $x, y \in V$, such that $G - \{x, y\}$ is disconnected. Then there is a component $C$ of $G - \{x, y\}$ such that the induced subgraph on $C \cup \{x, y\}$ with the edge $(x, y)$ is non-planar.

Proof

- $C_1, \ldots, C_k$ – the components of $G - \{x, y\}$.
- $G_i$ – the induced subgraph on $C_i \cup \{x, y\}$, plus the edge $(x, y)$.
- Assume for contradiction that $G_1, \ldots, G_k$ are all planar.
  - $H_1$ – a plane drawing of $G_1$.
  - $H_i$ (for $2 \leq i \leq k$) – drawing $G_i$ (without crossings) in a face of $H_{i-1}$ with $(x, y)$ on its boundary, and merging the two copies of $x, y$.
  - Each $H_i$ is planar, including $H_k = G$.

Contradiction!
Bad Math Joke #2

• **Q:** What do you get when you cross a mountain goat and a mountain climber?
• **A:** Nothing—you can’t cross two scalars.

3-Connectedness

• **Lemma.** Let $G = (V, E)$ be a graph with fewest edges among all non-planar graphs without a Kuratowski subgraph. Then $G$ is 3-connected.

• **Proof.**
  ◦ $G$ is a minimal non-planar graph.
  ◦ **By a previous lemma,** $G$ is 2-connected.
  ◦ We need to prove that there are no vertices $x, y \in V$ such that $G - \{x, y\}$ is disconnected.
Proof

- Assume for contradiction that there exist \( x, y \in V \) such that \( G - \{x, y\} \) is disconnected.
  - \( C_1, \ldots, C_k \): the components of \( G - \{x, y\} \).
  - By the previous lemma, there exists \( C_i \) such that the induced subgraph on \( C_i \cup \{x, y\} \) plus the edge \((x, y)\) is non-planar. Denote it as \( H \).
  - By the minimality of \( G \), \( H \) contains a Kuratowski subgraph \( K \).
  - Since \( G \) does not contain \( K \), it must be that \((x, y)\in K\) and \((x, y)\notin E\).

Proof (cont.)

- Let \( C' \) be another component of \( G - \{x, y\} \).
- In \( G \) there is a path \( P \) between \( x \) and \( y \) that uses only vertices of \( C' \).
- Combining \( P \) with the other edges of \( K \) yields a Kuratowski subgraph of \( G \). **Contradiction!**
Recap

- We proved that a smallest non-planar graph **without a Kuratowski subgraph** is **3-connected**.
  - To complete the proof of **Kuratowski's Theorem**, we prove that every 3-connected graph without a Kuratowski subgraph is **planar**.

Bad Math Joke #3

- **Q:** What do you get if you cross an elephant and a banana?
- **A:** |elephant| · |banana| · sin θ.
Contraction Cannot Generate Kuratowski Subgraphs

- **Lemma.** Let \( G = (V, E) \) be a graph with no Kuratowski subgraph. Then contracting any edge \( e \in E \) does not result in a Kuratowski subgraph.

- **Proof.**
  - \( G_e \) – the graph that is obtained by contracting \( e = (x, y) \) in \( G \).
  - Assume for contradiction that \( G_e \) contains a Kuratowski subgraph \( H \).

Proof

- \( v_e \) – vertex obtained by contracting \( e = (x, y) \).
- If \( v_e \) is not in \( H \), then \( H \) is also a subgraph of \( G \). **Contradiction!**
- \( v_e \) cannot have degree zero or one in \( H \).
- If \( v_e \) has degree two in \( H \), we can find \( H \) in \( G \) by replacing \( v_e \) with \( x \) and/or \( y \). **Contradiction!**
Proof (cont.)

- Consider the case where \( \text{deg } x \geq \text{deg } v_e \).
  - Then \( H \) is also in \( G \) with \( x \) replacing \( v_e \) and \( y \) being a subdivision vertex. **Contradiction!**

Proof (cont.)

- **A single case remains:** \( H \) is a subdivision of \( K_5 \) and after expanding \( e \) back both \( x \) and \( y \) are of degree 3.
  - In this case \( G \) contains \( K_{3,3} \). **Contradiction!**
  - In the figure, we have \( y, a, b \) on one side and \( x, c, d \) on the other.
Contractions and 3-Connectivity

- **Lemma.** Let \( G = (V, E) \) be a \textit{3-connected graph} with \( |V| \geq 5 \). Then there exists an edge \( e \in E \) whose \textit{contraction results in a 3-connected graph}.

\[
\text{Proof}
\]
- Assume \textit{for contradiction} that there exists a 3-connected \( G = (V, E) \) with \( |V| \geq 5 \), such that contracting any \( e \in E \) yields a graph \( G_e \) that is not 3-connected.
  - For any \( e \in E \), let \( v_e \) denote the vertex of \( G_e \) to which \( e \) is contracted.
  - Since \( G_e \) is not 3-connected, there exists \( z_e \in V \) such that \( G_e - \{v_e, z_e\} \) is disconnected.
Proof (cont.)

- Every $e = (x, y) \in E$ has $z_e \in V$ such that:
  - $G_e - \{v_e, z_e\}$ is disconnected.
  - $G - \{x, y, z_e\}$ is disconnected

- We choose an edge $e = (x, y)$ so that the size of the largest component $C$ of $G - \{x, y, z_e\}$ is maximized.
  - $C'$ — another component of $G - \{x, y, z_e\}$.
  - There must be an edge $f$ between $z_e$ and a vertex $u \in C'$.

Proof (cont.)

- Let $C'$ be another component. There is an edge $f$ between $z_e$ and a vertex $u \in C'$.
  - By definition, $G - \{z_e, u, z_f\}$ is disconnected.
  - The induced subgraph of $C \cup \{x, y\}$ is connected. Also, deleting $z_f$ from this subgraph cannot disconnect it, since this would imply that $G - \{z_e, z_f\}$ is disconnected (but $G$ is 3-connected!).
  - So $G - \{z_e, u, z_f\}$ is disconnected and contains a component larger than $C$.
  
  **Contradiction!**
The End

• A physicist and a mathematician are sitting in a faculty lounge. Suddenly, the coffee machine catches on fire. The physicist grabs a bucket and leaps toward the sink, fills the bucket with water, and puts out the fire.

• Another day, and the same two sit in the same lounge. Again the coffee machine catches on fire. This time, the mathematician stands up, gets a bucket, and hands the bucket to the physicist, thus reducing the problem to a previously solved one.