Recall: Plane Graphs

- A **plane graph** is a drawing of a graph in the plane such that the edges are non-crossing curves.
Recall: Planar Graphs

- The drawing on the left is not a plane graph. On the right we have a different drawing of the same graph, which is a plane graph.
- An abstract graph that can be drawn as a plane graph is called a planar graph.

Recall: Faces

- Given a plane graph, in addition to vertices and edges, we also have faces.
  - A face is maximal open (two-dimensional) region that is bounded by the edges.
  - There is always one unbounded face and any number of bounded faces.
Dual Graphs

- Every **connected** plane graph \( G = (V, E) \) has a **dual plane graph** \( G^* = (V^*, E^*) \).
  - Every vertex of \( V^* \) corresponds to a face of \( G \).
  - Every edge \( e^* \in E \) corresponds to an edge \( e \in E \). The edge \( e^* \) connects the vertices that are dual to the faces in the two sides of \( e \).

Euler’s Formula

- **Theorem.** Let \( G \) be a connected and not necessarily simple plane graph with \( v \) vertices, \( e \) edges, and \( f \) faces. Then
  \[ v + f = e + 2. \]

\[ v = 7 \]
\[ e = 10 \]
\[ f = 5 \]
Many Proofs

- In David Eppstein’s website, one can find 20 different proofs of Euler’s formula (see link on course’s webpage).

Proof 1: Induction

- We prove $v + f = e + 2$ by induction on $v$.
  - **Induction basis.** If $v = 1$ then the graph consists of $e$ loops.
  - The number of faces in the graph is $f = 1 + e$.
  - We indeed have $v + f = e + 2$. 

- Proof 1: Intersecting Trees
- Proof 2: Induction on Faces
- Proof 3: Induction on Vertices
- Proof 4: Induction on Edges
- Proof 5: Divide and Conquer
- Proof 6: Electrical Charge
- Proof 7: Dual Electrical Charge
- Proof 8: Sum of Angles
- Proof 9: Spherical Angles
- Proof 10: Pick’s Theorem
- Proof 11: Ear Decomposition
- Proof 12: Shelling
- Proof 13: Triangle Removal
- Proof 14: Noah’s Ark
- Proof 15: Binary Homology
- Proof 16: Binary Space Partition
- Proof 17: Valuations
- Proof 18: Hyperplane Arrangements
- Proof 19: Integer-Point Enumeration
- Proof 20: Euler tours
Induction Step

- We consider the case of \( v > 1 \).
  - Since \( G \) is connected, there exists an edge \((a, b)\) that is **not a loop**.
  - We **contract** \((a, b)\), but **without merging parallel edges**.
  - We obtain a graph \( G' \) with \( v - 1 \) vertices, \( e - 1 \) edges, and \( f \) faces.

\[ \text{Induction Step (cont.)} \]

- We consider the case of \( v > 1 \).
  - Since \( G \) is connected, there exists an edge \((a, b)\) that is **not a loop**.
  - We **contract** \((a, b)\), but **without merging parallel edges**.
  - We obtain a graph \( G' \) with \( v - 1 \) vertices, \( e - 1 \) edges, and \( f \) faces.
  - By the applying the **induction hypothesis** on \( G' \), we have \((v - 1) + f = (e - 1) + 2\).
  - Increasing both sides by 1 completes the proof.
Proof 2: Spanning Trees

- **Claim.** Consider a connected plane graph $G = (V, E)$. Let $T \subset E$ be the edges of a spanning tree of $G$. Then the edges dual to the edges in the set $S = E \setminus T$ form a spanning tree in $G^*$.  
  
- We do not prove the claim this year.

Completing the Proof

- We have
  
  ◦ $T$ – a spanning tree of $G$. Thus, it consists of $v - 1$ edges.
  ◦ $S^*$ – a spanning tree of $G^*$. Thus, it consists of $f - 1$ edges.
  ◦ $e = |S^*| + |T|$.
  ◦ Thus $e = (v - 1) + (f - 1)$. 
Maximal Plane Graphs

A planar graph $G$ is maximal if $G$ is simple and we cannot add another edge to $G$ without violating the planarity.

The Number of Edges in a Planar Graph

**Theorem.** A simple planar graph with $n \geq 3$ vertices has at most $3n - 6$ edges.

**Proof.**
- Consider a maximal planar graph $G$.
- $G$ must be connected, since otherwise we can add more edges to it.
- Consider a drawing of $G$ as a plane graph.
- Denote $e$ and $f$ as before and let $f_i$ be the length of the $i$’th face.
Proof

- Recall that \( \sum_i f_i = 2e \).
- Since every face is of length at least 3, we have \( 2e = \sum_i f_i \geq 3f \).
- Substituting this into \( n + f - 2 = e \) gives \( n + \frac{2}{3}e - 2 \geq e \rightarrow 3n - 6 \geq e \).

Back to \( K_5 \)

- **Problem.** Use what we learned today to provide an alternative proof for \( K_5 \) not being planar.
- **Answer.**
  - \( K_5 \) does not satisfy \( e \leq 3n - 6 \).
  - It has five vertices and \( 10 > 3 \cdot 5 - 6 \) edges.
Triangulations

- A **triangulation** is a simple plane graph such that **every face is of length three** (including the unbounded face).

Maximal Graphs are Triangulations

- **Claim.** Let $G = (V, E)$ be a (simple) plane graph with $|V| \geq 3$. Then the following are equivalent.
  - $|E| = 3|V| - 6$.
  - $G$ is a triangulation.
  - $G$ is a maximal plane graph.
Proof

- To prove $e \leq 3v - 6$ we combined
  - $v + f = e + 2$, and
  - $2e = \sum f_i \geq 3f$.
- The second inequality is sharp iff every face is of length three.
  - That is $e = 3v - 6$ iff every face is of length 3 (that is, a \textit{triangulation}).

Proof (cont.)

- If every face is of length three, then no edges can be added to the graph.
- If there is a face $F$ of length $> 3$, then the graph is not maximal since we can add a diagonal of $F$.
- Thus, a graph is \textit{maximal} iff it is a \textit{triangulation}.
Plane Graphs on a Sphere

- **Claim.** A graph can be drawn without crossings on a plane iff it can be drawn without crossings on a sphere.
  - Let $G$ be a graph drawn on the plane. If we take a sufficiently small patch $P$ of the sphere, it behaves like a plane, so $G$ can be drawn on it.

The Other Direction

- Let $G'$ be a graph drawn on the sphere. Let $N$ be a point in a face of $G'$. We consider the projection from $N$ onto a plane that is tangent to the sphere at the opposite point to $N$.
  - We obtain a drawing of the graph on the plane, where the face of $N$ becomes the unbounded face.
Polyhedrons

- A **polyhedron** is a solid object in a three-dimensional space, whose sides are flat two-dimensional polygons.
- A polyhedron $P$ is **convex** if for any two points $p, q \in P$ the straight-line segment $pq$ is fully contained in $P$.

Euler’s Formula for Polyhedrons

- Polyhedrons also have vertices, edges, and faces.
  - If the polyhedron is convex, then Euler’s formula $v + f = e + 2$ also applies to it.
  - We can “expand” a convex polyhedron so that its vertices would be on a sphere (we do not prove this rigorously).
  - Since we get a graph that is embedded on a sphere, we can embed it in the plane.
Platonic Solids

- In a regular polygon every edge has the same length.
- A Platonic solid is a convex polyhedron with all of its faces regular and congruent, and with all its vertices having the same degree.

Cube  Tetrahedron  Dodecahedron

History

- The ancient Greeks studied the Platonic solids extensively. Some sources credit Pythagoras with their discovery.
  - Pythagoras may have only been familiar with the tetrahedron, cube, and dodecahedron.
  - Theaetetus proved that there are exactly five.

Octahedron  Icosahedron
Graphs of the Platonic Solids

Using Euler’s Formula

- **Theorem.** There are exactly five Platonic solids.
- **Proof.** Consider such a solid, and a plane graph $G$ of it.
  - $k$ – the degree of every vertex.
  - $\ell$ – the length of every face.
  - By degree counting in $G$, we have $kv = 2e$.
  - By degree counting in $G^*$, we have $\ell f = 2e$.
  - Substituting these in Euler’s formula:
    \[
    \frac{2e}{k} + \frac{2e}{\ell} = e + 2
    \]
Proof

- We have
  \[ \frac{2e}{k} + \frac{2e}{\ell} = e + 2 \rightarrow e \left( \frac{2}{k} + \frac{2}{\ell} - 1 \right) = 2. \]

- The expression in the parenthesis must be positive, so
  \[ 2\ell + 2k - k\ell > 0 \rightarrow (k - 2)(\ell - 2) < 4. \]

- By definition, we have \( k, \ell \geq 3. \)

- The only values of \( k \) and \( \ell \) that satisfy the above are: \((3,3), (3,4), (4,3), (3,5), (5,3).\)

- Each pair uniquely determines one of the platonic solids.

The Corresponding Solids

- \( k \) – the degree of every vertex.
- \( \ell \) – the length of every face.

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<th>( \ell )</th>
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<th>( e )</th>
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[Images of the solids]
Fullerenes

- In chemistry, a **spherical fullerene** is a molecule of carbon atoms forming a hollow spherical “shell.” Many sizes and configurations are possible, but due to Carbon’s chemical properties they always have two common characteristics:
  - Each carbon atom bonds to 3 others.
  - The faces are rings of 5 or 6 carbon atoms.

The Math of Fullerenes

- **Theorem.** All Fullerenes have exactly 12 pentagons.
- **Proof.**
  - We think of the molecules as polyhedrons.
  - We have a plane graph that is 3-regular and has face lengths of five and six.
• $f_5$ – number of pentagons.
• $f_6$ – number of hexagons.
  ◦ $f = f_5 + f_6$.
  ◦ $e = \frac{5f_5 + 6f_6}{2}$.
  ◦ $v = \frac{2e}{3} = \frac{5f_5 + 6f_6}{3}$.
  ◦ Substitute these in Euler’s formula:
    $\frac{5f_5 + 6f_6}{3} + (f_5 + f_6) = \frac{5f_5 + 6f_6}{2} + 2$.
  ◦ Multiplying both sides by six:
    $(10f_5 + 12f_6) + (6f_5 + 6f_6) = (15f_5 + 18f_6) + 12$.
    $f_5 = 12$.

People Take Platonic Solids Seriously