Ma/CS 6b
Class 2: Matchings

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There aren’t enough crocodiles in the presentations

Is it OK to assume that P=NP??

Only today! 75% off for Morphine and Xanax.

Could you open every class by playing Flight of the Valkyries?
National Resident Matching Program

- Every **medical student** who is about to graduate ranks **hospitals** in which she wants to do her residency.
- Every **hospital** ranks **students** that it is interested in.
- Every year, over 20,000 applicants apply to about 1,800 programs.
- How can we handle this?

Reminder: Bipartite Graphs

- A graph $G = (V, E)$ is **bipartite** if we can partition $V$ into disjoint subsets $V_1, V_2 \subseteq V$ such that every edge of $E$ is between a vertex of $V_1$ and $V_2$.
- Equivalently, the vertices of $V$ can be colored red and blue such that no edge is **monochromatic**.
Reminder: Matchings

- A matching in a graph is a set of vertex-disjoint edges.
- The size of a matching is the number of edges in it.
- A maximum matching of $G$ is a matching of maximum size.

Back to the Medical Students

- How can we approach our medical students problem?
  - Bipartite graph – a vertex in $V_1$ for each student. A vertex in $V_2$ for each hospital.
  - An edge exists between a student and a hospital if they are interested in each other.
Solving the Problem?

- What should we do with the student-hospital graph?
  - We can find the maximum matching, but there are two problems with this.

First Problem

- **Problem.** Some hospitals might wish to take more than one resident.
- **Solution.** (as we saw in 6a)
  - If a hospital wants to take \( k \) residents, in the graph we have \( k \) vertices for it.
Second Problem

- Problem. We did not consider the rankings of the students and hospitals.
  - We might have chosen the red matching.
  - However, perhaps student A prefers hospital $\beta$, student $B$ prefers hospital $\alpha$, and similarly for the hospitals.

Alternating Paths

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, and let $M$ be a matching of $G$.
- A path is alternating for $M$ if it starts with an unmatched vertex of $V_1$ and every other edge of it is in $M$. 
Augmenting Paths

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, and let $M$ be a matching of $G$.
- A path is augmenting for $M$ if it is an alternating path of $M$, and ends in an unmatched vertex.

Using Augmenting Paths

- Consider a matching $M$ and an augmenting path $P$ of $M$.
- By switching in $P$ the edges that are in $M$ with the edges that are not, we obtain a larger matching.
Augmenting Paths and Matchings

• **Claim.** Let $G = (V_1 \cup V_2, E)$ be a bipartite graph and let $M$ be a matching in $G$. Then $M$ is **not a maximum matching** iff there exists an **augmenting path** for it.

• **Proof.**
  ◦ If there is an **augmenting path**, we can use it to **find a larger matching**, so $M$ is not a maximum matching.
  ◦ It remains to prove that when $M$ is not a maximum matching, there is an augmenting path for it.

Completing the Proof

• $M^*$ – a **maximum matching** of $G$.
• $F$ – the set of **edges** that are either in $M$ or in $M^*$, but not in both. Set $G' = (V, F)$.
• In $G'$, every vertex is of degree **at most two**.
• Thus, $G'$ is composed of **paths, cycles, and isolated vertices**. Since $|M| < |M^*|$, there must be at least one **augmenting path** for $M$. 
Traffic Cameras

• **Problem.** The city of Pasadena wants to have **traffic cameras that cover all of the roads of the city.**
  ◦ A camera covers 360° and sees far enough to cover a road at least until the next intersection.
  ◦ **How can we efficiently find the minimum number of cameras that are necessary?**

Considering the Problem as a Graph

• We build a graph:
  ◦ A **vertex** for every **intersection**.
  ◦ An **edge** between every two **adjacent intersections**.

• **What do we need to find in the graph?**
  ◦ A minimum set of vertices $S$ such that every edge is adjacent to at least one vertex of $S$. 
Vertex Covers

- Let $G = (V, E)$ be a graph. A **vertex cover** of $G$ is a set of vertices $V' \subset V$ such that every edge of $E$ is adjacent to at least one vertex of $V'$.

More About Vertex Covers

- **No polynomial-time algorithm is known** for finding the **minimum vertex cover**.
- A main open problem in Theoretical Computer Science.
  - Significantly easier in bipartite graphs.
König’s Theorem

• **Theorem.** Let \( G = (V_1 \cup V_2, E) \) be a bipartite graph. Then the size of a *maximum matching* of \( G \) is equal to the size of a *minimum vertex cover* of \( G \).

• **Proof.**
  ∘ \( M \) – a max matching.
  ∘ \( C \) – a min vertex cover.
  ∘ Since the edges of \( M \) are vertex-disjoint and \( C \) must contain a vertex of each, we have \( |C| \geq |M| \).

Proof (cont.)

• \( C \) – a min vertex cover.
• \( M \) – a max matching.
• We saw that \( |C| \geq |M| \).
• To complete the proof, it suffices to find a vertex cover of size \( |M| \).
• We build a *subset* \( V' \subseteq V \) by taking one vertex out of each edge \( e = (a, b) \) of \( M \).
  ∘ Take \( b \) if an *alternating path* of \( M \) ends in \( b \).
  ∘ Otherwise, take \( a \).
Proof (cont.)

- $V'$ consists of one vertex of each edge $(a, b) \in M$.
  - Take $b$ if an alternating path of $M$ ends in $b$.
  - Otherwise, take $a$.
- Assume for contradiction that an edge $(a, b) \in E$ is not covered by $V'$.
  - Either $a$ or $b$ must be matched in $M$, since otherwise $M$ is not a max matching.

The Case where $b$ is Matched

- Assume that $b$ is matched in $M$, but not $a$.
  - Then $(a', b) \in M$ for some $a' \in V_1$.
  - Since $b \notin V'$, we have $a' \in V'$ and no alternating path ends at $b$.
  - But $(a, b)$ is such an alternating path! Contradiction!
The Case where $a$ is Matched

- We proved that $a$ is matched in $M$.
  - $(a, b') \in M$ for some $b' \in V_2$.
  - Since $a \notin V'$, we have $b' \in V'$ and there is an alternating path $P$ ending at $b'$.
  - If $(a, b') \notin P$, then the path $P + (a, b') + (a, b)$ is an alternating path ending in $b$.
  - If $b$ is matched, then $b \in V'$ contradicting $(a, b)$ not being covered by $V'$.
  - If $b$ is unmatched, this an augmenting path for $M$, contradicting the maximality of $M$.

The Last Case

- It cannot be that $(a, b') \in P$. No alternating path can end in $(a, b')$.
  - In the path, we move from $V_1$ to $V_2$ only with unmatched edges.
Concluding the Proof

- $M$ – a maximum matching.
- We defined a subset $V' \subset V$ of size $|M|$ and proved that it is a vertex cover.
- We also proved that any vertex cover is of size at least $|M|$, implying that $V'$ is a minimum vertex cover.
  - That is, the minimum vertex cover has the same size as the maximum matching.

Vertex Covers in Bipartite Graphs

- **Problem.** Describe an efficient algorithm for finding a min vertex cover in a bipartite graph $G = (V_1 \cup V_2, E)$.
- **Solution.**
  - From 6a, we know an algorithm for finding a maximum matching $M$ in a bipartite graph.
  - We pick one vertex out of each edge $(a, b) \in M$. Take $b$ if an alternating path ends in $b$. Otherwise, take $a$.
  - But how do we know whether such a path exists?
Finding an Alternating Path

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, and let $M$ be a max matching.
- We wish to find whether there is an alternating path for $M$ ending at $b \in V_2$.
  - We run a variant of BFS from $b$.
  - We already did this in detail in 6a.

The End