1. Let $G = (V, E)$ denote a graph that does not contain a perfect matching, even though $|V|$ is even. Thus, there exists a set $S \subset V$ violates Tutte’s condition. Let $n = |V|$ and $k = |S|$. Find the maximum number of edges that $G$ can have, as a function of $n$ and $k$. Make sure to find the exact value for every $k$ and to briefly explain why it is indeed the maximum. You may assume that $n$ is much larger than $k$.

2. Consider a connected graph $G = (V, E)$ such that $|V|$ is even and $G$ does not contain $K_{1,3}$ as an induced subgraph.\footnote{$K_{s,t}$ is a “complete” bipartite graph with $s$ vertices on one side, $t$ vertices on the other side, and all of the $st$ edges between the two sides.} Prove that $G$ contains a perfect matching by showing that it satisfies Tutte’s condition (do not find a perfect matching by using a different method).

3. Find the smallest 3-regular graph that has a connectivity of 1. Explain why a smaller graph with these properties cannot exist.

4. Let $G = (V, E)$ be a connected graph with the following special property: For every $e \in E$, there exist two cycles $C_1, C_2$ in $G$ that have exactly one edge in common — the edge $e$. The two cycles may have more vertices in common, and there may be additional cycles that contain $e$.

   Prove or disprove: The graph $G$ must be 3-edge-connected.