1. (a) Suppose \( A = B \oplus C \). For any \( x \in U \), \( x = \sum_{i=1}^{n} f_i x_i \) where \( f_{m+1} = \cdots = f_n = 0 \), so \( Ax \) is also in \( U \). Hence \( \alpha(U) \subset U \). Similarly one sees that \( \alpha(W) \subset W \). The characterizations of \( B, C \) are clear from the form of \( A \).

For the converse, notice that since \( \alpha(U) \subset U \), \( \alpha(W) \subset W \) and \( U \cap W = \{0\} \), the entries of \( A = (a_{ij}) \) must satisfy \( a_{ij} = 0 \) whenever \( i \leq m \) and \( j > m \), or \( i > m \) and \( j \leq m \). Thus \( A = B \oplus C \).

(2) By the determinant formula for block matrices, we know \( m_A = \det(xI - A) = \det(xI - B)\det(xI - C) = m_Bm_C \). Because of the form of \( A \), we know that \( \text{Ann}(A) = \text{Ann}(B) \cap \text{Ann}(C) \), so \( (m_A) = (m_B) \cap (m_C) \), but \( F[x] \) is a PID, so \( m_A = \text{lcm}(m_B, m_C) \).

(3) \( A \) is diagonalizable if and only iff \( m_A \) factors into distinct linear factors over \( F \), which by part (2) is equivalent to \( m_B, m_C \) each factor into distinct linear factors over \( F \), which in turn is equivalent to \( B, C \) being diagonalizable.

2. Looking at factorizations of the given cyclic modules, one sees that over \( \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) the invariant factors must be \( x^4 - 16, (x - 2)^2(x + 2)(x^2 + 4) \). Over \( \mathbb{Q} \) and \( \mathbb{R} \), \( x^2 + 4 \) is irreducible, so the elementary divisors are \( (x - 2), (x - 2)^2, (x + 2), (x + 2), (x^2 + 4) \) and \( (x^2 + 4) \). Over \( \mathbb{C} \), we have \( (x - 2), (x - 2)^2, (x + 2), (x + 2), (x + 2i), (x + 2i), (x - 2i), (x - 2i) \) and \( (x - 2i) \).

3. 3-by-3 matrices have characteristic polynomials of degree 3, so the only possible characteristic polynomials are \((x + 1)^2(x - 1)\) and \((x + 1)(x - 1)^2\), which correspond to similarity classes with representatives \( \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \), respectively.

4. The Jordan canonical form only exists when \( F \) contains all the eigenvalues of the matrix in question, so we assume \( F \) is an extension of \( \mathbb{Q} \) which contains \( i \). There are eight possible cases:

1. invariant factor: \((x + 1)^2(x^2 + 3)^2\);
2. invariant factors: \((x + 1)(x^2 + 3), (x + 1)(x^2 + 3)\);

\[
\begin{align*}
\text{RCF} &= \begin{pmatrix} 1 & -3 \\ 1 & -1 \end{pmatrix} ; \\
\text{JCF} &= \begin{pmatrix} -1 & 1 \sqrt{3} \\ -1 & -1 \sqrt{3} \end{pmatrix}
\end{align*}
\]

3. invariant factors: \((x + 1)^2(x^2 + 3), (x^2 + 3)\);

\[
\begin{align*}
\text{RCF} &= \begin{pmatrix} 1 & -3 \\ 1 & -1 \end{pmatrix} ; \\
\text{JCF} &= \begin{pmatrix} -1 & 1 \sqrt{3} \\ -1 & -1 \sqrt{3} \end{pmatrix}
\end{align*}
\]

4. invariant factors: \((x + 1)(x^2 + 3)^2, x + 1\);

\[
\begin{align*}
\text{RCF} &= \begin{pmatrix} 1 & -9 \\ 1 & -6 \end{pmatrix} ; \\
\text{JCF} &= \begin{pmatrix} -1 & 1 \sqrt{3} \\ -1 & -1 \sqrt{3} \end{pmatrix}
\end{align*}
\]

5. invariant factors: \((x + 1)(x + i \sqrt{3}), (x + 1)(x + i \sqrt{3})(x - i \sqrt{3})^2\);

\[
\begin{align*}
\text{RCF} &= \begin{pmatrix} 1 & 3i \sqrt{3} \\ 1 & i \sqrt{3} - 1 \end{pmatrix} ; \\
\text{JCF} &= \begin{pmatrix} -1 & 1 \sqrt{3} \\ -1 & -1 \sqrt{3} \end{pmatrix}
\end{align*}
\]

6. invariant factors: \((x + 1)(x - i \sqrt{3})(x + 1)(x + i \sqrt{3})^2(x - i \sqrt{3})\);

\[
\begin{align*}
\text{RCF} &= \begin{pmatrix} i \sqrt{3} & 3i \sqrt{3} \\ i \sqrt{3} - 1 & i \sqrt{3} - 1 \end{pmatrix} ; \\
\text{JCF} &= \begin{pmatrix} -1 & 1 \sqrt{3} \\ -1 & -1 \sqrt{3} \end{pmatrix}
\end{align*}
\]

7. invariant factors: \(x - i \sqrt{3}, (x + 1)^2(x - i \sqrt{3})(x + i \sqrt{3})^2\);
8. invariant factors: \( x + i\sqrt{3}, (x + 1)^2(x + i\sqrt{3})(x - i\sqrt{3})^2; \)

\[
\begin{pmatrix}
-i\sqrt{3} & 3i\sqrt{3} \\
1 & 6i\sqrt{3} - 3 \\
1 & 4i\sqrt{3} - 6 \\
1 & 2i\sqrt{3} - 4 \\
1 & i\sqrt{3} - 2
\end{pmatrix}; 

\begin{pmatrix}
-1 & 1 \\
-1 & i\sqrt{3} \\
i\sqrt{3} & 1 \\
i\sqrt{3} & -i\sqrt{3}
\end{pmatrix}
\]

5. By the rational root test one sees that \( f(T) = T^3 + 5T + 5 \) is irreducible in \( \mathbb{Q}[x] \), so \( f(T) \) is the minimal polynomial of \( T \), so \( \text{char}(T) = f(T)^n \). But \( \dim(V) = \deg(\text{char}(T)) \), so \( 3|\dim(V) \). When \( \dim(V) = 3 \), \( \text{char}(T) = f(T) \), so there is only one similarity class, of which a representative is

\[
\begin{pmatrix}
0 & 0 & -5 \\
1 & 0 & -5 \\
0 & 1 & 0
\end{pmatrix}.
\]