10.1.5. Clearly $IM \subset M$. $IM \neq \emptyset$ since $0 \cdot 0 = 0 \in IM$. For any $\sum a_i m_i, \sum b_j m'_j \in IM$ and any $r \in R$, we have $\sum a_i m_i + r \cdot \sum b_j m'_j = \sum a_i m_i + \sum (rb_j)m'_j \in IM$. Thus $IM$ is a submodule of $M$.

10.1.8. (a) Suppose $R$ is an integral domain. $0 \in Tor(M)$ since $r \cdot 0 = 0$ for any $r \in R$, so $Tor(M)$ is nonempty. Let $a, b \in Tor(M)$, then $xa = yb = 0$ for some nonzero $x, y \in R$. But $R$ is an integral domain, so $xy \neq 0$. Then for any $r \in R$, $xy(a + rb) = y(xa) + rx(yb) = 0 + 0 = 0$, so $a + rb \in Tor(M)$. Thus $Tor(M)$ is a submodule of $M$.

(b) Let $R = \mathbb{Z}/10\mathbb{Z}$ and consider $M = R$ as an $R$-module over itself. Then $2, 5 \in Tor(M)$, but $2 + 5 = 7 \notin Tor(M)$, so $Tor(M)$ is not a submodule.

(c) Suppose $R$ has zero-divisors and let $M$ be a nontrivial $R$-module. Let $r \in R$ be a zero divisor, then $rs = 0$. Let $m \in M$ be a nonzero element, then $rsm = 0m = 0$, so $sm \in Tor(M)$. If $sm \neq 0$, $sm$ is a nonzero element of $Tor(M)$. If $sm = 0$, then $m \in M$ is a nonzero element.

10.2.8. Let $m \in Tor_R(M)$, then there exists some $r \in R$ such that $rm = 0$. Then $r\phi(m) = \phi(rm) = \phi(0) = 0$, so $\phi(m) \in Tor_R(N)$. Therefore $\phi(Tor_R(M)) \subset Tor_R(N)$.

10.2.13. Because $\phi$ is surjective, we have $N/IN = \overline{\phi}(M/IM) = (\phi(M) + IN)/IN$ (The last equality follows from the definition of the induced map), so $N = \phi(M) + IN$ by the fourth isomorphism theorem. Since $I$ is nilpotent, $I^k = 0$ for some nonnegative integer $k$. Substituting the expression into itself $k$ times, we see $N = \phi(M) + I(\phi(M) + IN) = \phi(M) + I\phi(M) + I^2N = \phi(M) + \cdots = \phi(M) + I^kN = \phi(M)$. Hence $\phi$ is surjective.

10.3.5. Let $m_1, \cdots, m_n$ be a set of generators of $M$. Since $M$ is torsion, for each $i = 1, \cdots, n$ there exist some nonzero $r_i \in R$ such that $r_im_i = 0$. Now $R$ is an integral domain, so $a_1 \cdots a_n \neq 0$. Let $x$ be any element of $M$, then $x = \sum_{i=1}^n b_im_i$ for some $b_i \in R$. In particular, $R$ is commutative, so we have $a_1 \cdots a_n \cdot x = a_2 \cdots a_nb_1(a_1m_1) + \cdots + a_1 \cdots a_{n-1}b_{n-1}(b_nm_n) + 0 + \cdots + 0 = 0$, so $a_1 \cdots a_n \in Ann_R(M)$ and $Ann_R(M)$ is nontrivial.

For an example, consider $\mathbb{Q}/\mathbb{Z}$ as a $\mathbb{Z}$-module. If $Ann_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z})$ is nontrivial, it contains some nonzero element $x$. Let $y \in \mathbb{Z}$ be an element that does not divide $x$, then $0 = x \cdot \frac{1}{y} = \frac{x}{y}$, so $x/y$ is an integer, which implies $y|x$, a contradiction. Thus $Ann_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) = 0$.

10.3.18. We first check that $M = \sum Ann_M(p_i^{k_i})$. For any $j$, write $q_j = \prod_{i \neq j} p_i^{k_i}$. For any $q_j \cdot m \in$
(q_j)M we have $p_j^{k_j}q_jm = am = 0$, so $q_j \cdot m \in \text{Ann}_M(p_j^{k_j})$, so $(q_j)M \subset \text{Ann}_M(p_j^{k_j})$. On the other hand, for any $m \in \text{Ann}_M(p_j^{k_j})$, there exist $x, y \in R$ such that $1 = q_jx + p_j^{k_j}y$ since $(q_j, p_j) = 1$ by assumption. Thus $m = (q_jx + p_j^{k_j}y)m = q_jxm + p_j^{k_j}ym = q_jxm \in (q_j)M$, so $\text{Ann}_M(p_j^{k_j}) \subset (q_j)M$. Thus $\text{Ann}_M(p_j^{k_j}) = (q_j)M$. Note that by the definition of the $q_j$ we know $(q_1, \cdots, q_n) = R$, so $\sum r_iq_i = 1$ for some $r_i \in R$. Thus for any $m \in M$, we have $m = \sum r_iq_im \in \sum (q_i)M$.

It remains to show that the sum is direct, i.e. $(q_j)M \cap \sum_{i \neq j} (M) = 0$ for all $j$. Suppose $m \in (q_j)M \cap \sum_{i \neq j} (M)$. Since $q_j$ and $p_j^{k_j}$ are coprime, there exist $x, y \in R$ such that $xp_j^{k_j} + yq_j = 1$. Moreover, $p_j^{k_j}(q_j)M = 0$ and $q_j \sum_{i \neq j} (M) = 0$, so $m = (xp_j^{k_j} + yq_j)m = xp_j^{k_j}m + yq_jm = 0$. Thus we are done.

**Additional Problem.**  (1) Let $n, n' \in N[I]$. Then $a (n + n') = an + an' = 0$, so $n + n' \in N[I]$ and $N[I]$ is closed under addition.

(2) Define a map $\phi : \text{Hom}_R(M, N) \to N[\text{Ann}(m)]$ by $f \mapsto f(m)$. This is well-defined because for any $x \in \text{Ann}(m)$ we have $af(m) = f(am) = 0$, which implies $f(m) \in N[\text{Ann}(m)]$. $\phi$ is a homomorphism because $\phi(f + g) = (f + g)(m) = f(m) + g(m) = \phi(f) + \phi(g)$ for any $f, g \in \text{Hom}_R(M, N)$. Since $M$ is cyclic, any homomorphism from $M$ to $N$ is uniquely determined by where it sends the generator, so $\phi$ is injective. Finally, $\phi$ is clearly surjective. Thus we have the desired isomorphism of abelian groups.

(3) Let $\phi \in \text{End}_R(M)$ be a nontrivial element, consider $\text{ker}\phi$. Since $M$ is irreducible, $\text{ker}\phi = 0$ or $M$. But $\phi$ is nontrivial, so $\text{ker}\phi = 0$, which implies $\text{img}\phi = M$. Thus $\phi$ is bijective and admits an inverse. Hence $\text{End}_R(M)$ is a division ring.

Now assume $R$ is commutative.

(1) Note that $N[I] \neq \emptyset$ since $0 \in N[i]$. For any $n, n' \in N[I]$ and $r \in R$ we have $a(n + rn') = an + ran' = 0$ for any $a \in I$, so $x + ry \in N[I]$. Thus $N[I]$ is a sub-$R$-module of $N$.

(2) We already showed isomorphism as abelian groups, so it suffices to check that $\phi$ is a homomorphism of $R$-modules. For any $r \in R$ and $f \in \text{Hom}_R(M, N)$ we have $\phi(rf) = (rf)(m) = r(f(m)) = r\phi(f)$, so we conclude the desired isomorphism of $R$-modules holds.

(3) We already showed that $\text{End}_R(M)$ is a division ring. It suffices to show that $F \subset Z(\text{End}_R(M))$, because any commutative division ring is a field. Consider the map $\psi : R \to \text{End}_R(M)$ given by $r \mapsto (m \mapsto rm)$, then $\psi$ is clearly a ring homomorphism with kernel $\text{Ann}(m)$. Thus $\text{img}\psi \cong R/\text{Ann}m = F$ and we may identify any element in $F$ with an element in $\text{img}\psi$. For any $\psi(r) \in \text{img}\psi$ and any $f \in \text{End}_R(M)$, we have $(\psi(r) \circ f)(m) = rf(m) = (f)(rm) = (f \circ \psi(r))(m)$ for any $m \in M$. Thus $F \subset Z(\text{End}_R(M))$ and we are done.