Caltech Department of Mathematics

Ma 3/103 KC Border
Introduction to Probability and Statistics Winter 2017

Assignment 7: Confidence Intervals and Parametric Hypothesis Testing

Due Tuesday, February 28 by 4:00 p.m. at 253 Sloan

Instructions:

When asked for a confidence interval or a probability, give both a formula and an explanation for why you used that formula, and also give a numerical value when available.

When asked to plot something, use informative labels (even if handwritten), so the TA knows what you are plotting, attach a copy of the plot, and, if appropriate, the commands that produced it.

No collaboration is allowed on optional exercises.

Exercise 1 (Confidence Intervals) (45 pts)

For this question, the calculations are trivial. What matters is your reason for doing them. You *must* explain and defend your reasoning.

When we discussed confidence intervals for the mean of a normal distribution with unknown mean μ and known standard deviation σ, based on a sample of n independent random variables, we started with the maximum likelihood estimate $\hat{\mu}$, showed that $(\hat{\mu} - \mu)/(\sigma/\sqrt{n})$ is a standard normal, and identified an interval $[a, b]$ so that $P_{(\mu,\sigma^2)}\left((\hat{\mu} - \mu)/(\sigma/\sqrt{n}) \in [a, b]\right) = 1 - \alpha$. We then found an interval $[\bar{a}(\hat{\mu}), \bar{b}(\hat{\mu})]$ so that

$$\frac{\hat{\mu} - \mu}{\sigma/\sqrt{n}} \in [a, b] \iff \mu \in [\bar{a}(\hat{\mu}), \bar{b}(\hat{\mu})],$$

and called that interval the $1 - \alpha$ confidence interval for μ. Just to make sure we're on the same page,

1. (4 pts) what are the expressions for $\bar{a}(\hat{\mu})$ and $\bar{b}(\hat{\mu})$? (Hint: they depend on n, $\hat{\mu}$, $z_{\alpha/2}$, and σ.)

Now consider the continuous "German tank problem." Here the probability model is

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} & 0 \leqslant x \leqslant \theta \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 0$ is the unknown parameter. The data comprise a sample of n independent draws $\boldsymbol{x} = (x_1, \ldots, x_n)$ from this distribution.

2. (5 pts) What is the maximum likelihood estimator $\hat{\theta}$ of θ, written as a function of the datum $\boldsymbol{x}$?

Okay, here comes the real question: How would you construct a $1 - \alpha$ confidence interval for θ? It helps to break down your answer into steps:

3. (10 pts) If you knew θ, what is the smallest interval $[a, b]$ (the endpoints may depend on θ) such that
$$P_\theta\left(\hat{\theta}(\boldsymbol{x}) \notin [a, b]\right) = \alpha?$$

4. (10 pts) Find an interval $[\bar{a}(\hat{\theta}), \bar{b}(\hat{\theta})]$ such that
$$\left(\hat{\theta} \notin [a, b]\right) \iff \left(\theta \notin [\bar{a}(\hat{\theta}), \bar{b}(\hat{\theta})]\right).$$

5. (4 pts) Compute $\bar{a}(\hat{\theta}), \bar{b}(\hat{\theta})$ for $n = 5$, and $\alpha = 0.10, 0.05, 0.01$.

6. (10 pts) Derive the method of moments estimator of θ.

7. (4 pts) Must the method of moments estimate fall in the confidence interval? (Why or why not?)

□

Exercise 2 (20 pts)

I love hardware stores. One of the things you can now find at a decent hardware store is a laser "tape measure." It is battery powered and has digital readout of the length. The length is measured with an error e that has mean zero and small variance σ^2. You have two boards, of lengths A and B, where A is obviously greater than B. You may make two measurements before the battery in your laser tape measure dies. Errors on different measurements are stochastically independent.

You could (i) measure each board separately. Or you could (ii) align the boards on one end and measure the difference $A-B$ of their lengths and then put them end-to-end and measure the sum $A+B$ of their lengths.

1. In case (i), what is the variance of the measurement of A? What is the variance of the measurement of B?

2. In case (ii), let D be the measurement of the difference in lengths $A-B$, and let S be the measurement of the sum of lengths $A+B$. Then an estimate of the length of A is $a = (S+D)/2$, and an estimate of the length of B is $b = (S-D)/2$. What is the variance of the measurement of a? What is the variance of the measurement of b?

3. What is the better method for ascertaining the lengths?

□

The following problems are taken from Larsen and Marx [2]. I have slightly modified the text of some of the questions. These problems are not intellectually challenging, but they are representative of the sort of routine hypothesis testing that every researcher needs to carry out in order to publish their work. (It's kind of like practicing scales on a musical instrument. It's not fun [for most people], but you want to be able to do it without using your higher brain functions. Legend has it that Charlie Parker spent eight to sixteen hours a day for five years practicing scales in a woodshed. It's unlikely we would have bebop if he hadn't.)

Exercise 3 (25 pts) [2, Problem 7.4.9, pp. 399–400]:

Creativity, as any number of studies have shown, is very much a province of the the young. Whether the focus is music, literature, science, or mathematics an individual's best work seldom occurs late in life. Einstein, for example, made his most profound discoveries at the age of twenty-six; Newton, at the age of twenty-three. Robert Wood [3] compiled the following list of twelve scientific breakthroughs dating from the middle of the sixteenth century to to the early years of the twentieth century. All represented high-water marks in the careers of the scientists involved.

Discovery	Discoverer	Year	Age, y
Earth orbits sun	Copernicus	1543	40
Telescope, basic laws of astronomy	Galileo	1600	34
Principles of motion, gravity, calculus	Newton	1665	23
Nature of electricity	Franklin	1746	40
Burning is oxidation	Lavoisier	1774	31
Earth evolved by gradual processes	Lyell	1839	33
Natural selection and evolution	Darwin	1858	49
Field equations for light	Maxwell	1864	33
Radioactivity	Curie	1896	34
Quantum theory	Planck	1901	43
Special relativity, $e = mc^2$	Einstein	1905	26
Quantum wave equation	Schrödinger	1926	39

1. (10 pts) What are the sample average and standard deviation? What can be inferred from these data about the *true* age at which scientists do their best work? Answer the question by constructing a 95% confidence interval.

2. (5 pts) Before constructing a confidence interval for a set of observations over a long time period, we should be convinced the y_is exhibit no biases or trends. If, for example, the age at which scientists made major discoveries decreased from century to century, the the parameter μ would no longer be a constant, and the confidence interval would be meaningless. Plot the age versus date for these twelve discoveries. (Put the date on the abscissa.) Does the variability of the y_is appear to be random with respect to time?

3. (10 pts) Give a good reason for questioning how much light these data shed on the age of greatest scientific creativity. Hint: How do you think the sample was constructed? How would you have picked your sample?

□

Exercise 4 (15 pts) Problem 6.4.8, p. 378.

For a normal probability model, will $n = 45$ be a sufficiently large sample to test $H_0\colon \mu = 10$ versus $H_1\colon \mu \neq 10$ at the $\alpha = 0.05$ level of significance if the experimenter wants the Type II error probability to be no greater than 0.20 when $\mu = 12$? Assume that $\sigma = 4$.

□

Exercise 5 (20 pts) Problem 6.4.18, p. 379. An experimenter takes a sample of size 1 from the Poisson probability model, $p_X(k) = e^{-\mu}\mu^k/k!$, $k = 0, 1, 2, \dots$, and wishes to test

$$H_0\colon \mu = 6$$
against
$$H_1\colon \mu < 6.$$

by rejecting H_0 if $k \leqslant 2$.

1. Calculate the probability of committing a Type 1 error.

2. Calculate the probability of committing a Type II error when $\mu = 4$.

□

Exercise 6 (10 pts) How much time did you spend on the previous exercises? □

Exercise 7 (Optional Exercise) (50 pts) The set A has n elements, and so has 2^n subsets. The subsets are placed into an urn, and m subsets $E_1, \dots, E_m$ are drawn in order at random with replacement from the urn.

What is the probability that

$$E_1 \subseteq E_2 \subseteq \cdots \subseteq E_m \text{ ?}$$

□

References

[1] R. L. Collins. 1968. On the inheritance of handedness. *Journal of Heredity* 59(1):9–12. http://jhered.oxfordjournals.org/content/59/1/9.full.pdf+html

[2] R. J. Larsen and M. L. Marx. 2012. *An introduction to mathematical statistics and its applications*, fifth ed. Boston: Prentice Hall.

[3] R. M. Wood. 1970. Giant discoveries of future science. *Virginia Journal of Science* 21:169–177.