2. (20 pts) [Ch. 4, Problem 2.5 (a), (c)]

Solution. Let $A, C$ be the matrices in question:

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -2 & 2 & 6 \\ 5 & 1 & -6 \\ -5 & 2 & 9 \end{pmatrix}$$

(a) We have $\det(A - \lambda I) = (4 - \lambda)(1 - \lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$. There are two distinct eigenvalues, so we know $A$ must be diagonalizable. Some standard computations give us

$$\ker(A - 2I) = \text{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \ker(A - 3I) = \text{span} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$ 

Thus $\{(1, 1)^T, (2, 1)^T\}$ forms our basis of eigenvectors, and we have the diagonalization

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1}.$$ 

(c) Let us compute the characteristic polynomial of $C$:

$$\det(C - \lambda I) = \det \begin{pmatrix} -2 - \lambda & 2 & 6 \\ 5 & 1 - \lambda & -6 \\ -5 & 2 & 9 - \lambda \end{pmatrix} = \det \begin{pmatrix} 3 - \lambda & 3 - \lambda & 0 \\ 5 & 1 - \lambda & -6 \\ -5 & 2 & 9 - \lambda \end{pmatrix}$$

$$= (3 - \lambda) \left( \det \begin{pmatrix} 1 - \lambda & -6 \\ 2 & 9 - \lambda \end{pmatrix} - \det \begin{pmatrix} 5 & -6 \\ -5 & 9 - \lambda \end{pmatrix} \right)$$

$$= (3 - \lambda) \det \begin{pmatrix} -4 - \lambda & -6 \\ 7 & 9 - \lambda \end{pmatrix} = (3 - \lambda)(\lambda^2 - 5\lambda + 6) = (3 - \lambda)^2(2 - \lambda).$$

Notice that we added the second row to the first before performing cofactor expansion along the first row, and combined the resulting two $2 \times 2$ determinants into one via linearity in the last column. We have then $\lambda = 3$ with algebraic multiplicity 2 and $\lambda = 2$ once. Some row reductions later we arrive at

$$\ker(A - 2I) = \text{span} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \ker(A - 3I) = \text{span} \begin{pmatrix} 2 \\ 5 \\ 6 \\ 0 \end{pmatrix}.$$ 

Since the eigenspace for $\lambda = 3$ has dimension 2, we can diagonalize $C$. We now do so:

$$C = \begin{pmatrix} 1 & 2 & 6 \\ -1 & 5 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 6 \\ -1 & 5 & 0 \\ 1 & 0 & 5 \end{pmatrix}.$$ 

□

Date: February 27, 2017.
3. (20 pts) [Ch. 4, Problem 2.6] Consider the matrix

$$A = \begin{pmatrix} 2 & 6 & -6 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{pmatrix}$$

a) Find its eigenvalues. Is it possible to find the eigenvalues without computing?

b) Is this matrix diagonalizable? Find out without computing anything.

c) If the matrix is diagonalizable, diagonalize it.

**Solution.**
a) Indeed it is possible to find the eigenvalues of $A$ without computing anything. Note that $A$ is an upper triangular matrix and, consequently, so is $A - \lambda I$ for indeterminate $\lambda$. Taking the determinant, we note that the determinant of an upper triangular matrix is the product of its diagonal elements. Thus, the characteristic polynomial of $A$ is

$$p_A(\lambda) = \det(A - \lambda I) = - (\lambda - 2)(\lambda - 5)(\lambda - 4).$$

Consequently, the eigenvalues of $A$ are 2, 4, and 5.

b) The matrix $A$ is indeed diagonalizable. This is because it is a $3 \times 3$ matrix with 3 distinct eigenvalues and, by Corollary 2.3 in chapter 4, is thus diagonalizable.

c) After a bit of algebra, we find a set of eigenvectors corresponding to the set of $A$ to be

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \quad \text{and } v_5 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$ 

Noting that these form a basis, we may tabulate these vectors into an invertible matrix

$$Q = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

with inverse $Q^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$.

Finally, taking our diagonal matrix consisting of eigenvalues

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

we see that

$$A = \begin{pmatrix} 2 & 6 & -6 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} = QDQ^{-1}$$

\[ \square \]

4. (20 pts) Let $T : \mathbb{P}_n \to \mathbb{P}_n$ be the linear operator given by $(Tf)(t) = f(-t)$. Find the eigenvalues and eigenvectors of $T$.

**Solution.** Note that the space $\mathbb{P}_n$ of polynomials of degree less than or equal to $n$ is of dimension $n+1$ and has as a basis the monomials $1, t, t^2, \ldots, t^n \in \mathbb{P}_n$. Furthermore, operating on each such monomial, one has that $Tt^k = (-t)^k = (-1)^k t^k$. Thus, each of the $t^k$ is an eigenvector of $T$ with eigenvalue $(-1)^k$. Thus, as $1, t, \ldots, t^n$ is a basis, it is an eigenbasis, and the only eigenvalues of $T$ are therefore $\pm 1$, or $\sigma(T) = \{1, -1\}$. Thus, $T$ splits $\mathbb{P}_n$ into two eigenspaces, the 1-eigenspace consisting of even polynomials (polynomials made up of linear combinations of only even degree
monomials), and the \((-1)\)-eigenspace consisting of odd polynomials (polynomials made up of linear combinations of only odd degree monomials).

5. (20 pts) A linear map \( T : V \to V \) is called nilpotent if \( T^n = 0 \) for some \( n \). Let \( V \) be a finite dimensional vector space over \( F \) and \( T : V \to V \) be a non-zero nilpotent operator.
   (1) Find all the possible eigenvalues of \( T \).
   (2) Show that \( I + T \) is invertible.
   (3) Is \( I + T \) diagonalizable?

Solution. (1) Suppose \( v \) is an eigenvector of \( T \), with eigenvalue \( \lambda \). Since \( T^n = 0 \), we have
\[
0 = T^n v = \lambda^n v.
\]
But \( v \) is nonzero, as it is an eigenvector. Thus \( \lambda^n = 0 \), so \( \lambda = 0 \); the only possible eigenvalue for \( T \) is 0.

(2) If \( v \) is an eigenvector of \( T + I \) with eigenvalue \( \lambda \), we see that
\[
Tv = ((T + I) - I)v = (\lambda - 1)v,
\]
and so \( v \) is an eigenvector of \( T \) with eigenvalue \( \lambda - 1 \). Therefore \( \lambda - 1 = 0 \), and \( \lambda = 1 \). We conclude that the only possible eigenvalue for \( T + I \) is 1; in particular 0 cannot be an eigenvalue of \( T + I \), and so it has full rank and is hence invertible.

(3) Let us assume \( T + I \) is diagonalizable. Its only eigenvalue is 1, which must then occur with algebraic and geometric multiplicity \( \dim V \). Thus the eigenspace corresponding to \( \lambda = 1 \) is all of \( V \). In other words, \( (T + I)v = v \) for all \( v \in V \). We must then have \( T + I = I \), in which case \( T = 0 \). Thus we see that for a nonzero nilpotent \( T \), \( T + I \) is not diagonalizable.

6. (20 pts) Let \( V = \mathbb{P}^3 \) be the vector space of degree at most 3 polynomials in one variable \( x \) (with complex coefficients). Let \( T \) be the linear operator \( T(f) = xf' + f'' \).
   (You don’t need to check that \( T \) is linear.) (a) Calculate the eigenvalues of \( T \).
   (b) For each eigenvalue, find a basis of the corresponding eigenspace.
   (c) Give a basis of \( V \) for which \( T \) is represented by a diagonal matrix.

(Hint: Observe that what you have done amounts to finding the values of the parameter \( \lambda \) for which the differential equation \( \lambda f - xf' - f'' = 0 \) has a polynomial solution, and for such values finding the solutions.)

Solution. (a) We need to find \( \lambda \) such that \( T(f) = \lambda f \) for some non-trivial \( f \). As a result, we need to find \( \lambda \) such that \( \lambda f - xf' - f'' = 0 \). If we write \( f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \), then the equation \( \lambda f(x) - xf'(x) - f''(x) = 0 \) gives us
\[
(\lambda - 3)a_3 x^3 + (\lambda - 2)a_2 x^2 + (\lambda a_1 - a_1 - 6a_3)x + \lambda a_0 - 2a_2 = 0,
\]
so we get:
\[
\begin{align*}
(\lambda - 3)a_3 &= 0 \\
(\lambda - 2)a_2 &= 0 \\
(\lambda - 1)a_1 &= 6a_3 \\
\lambda a_0 &= 2a_2
\end{align*}
\]
To solve the above, we need to condition on whether the coefficients of \( f \) are zero, so we have the following cases:
• if $a_3 \neq 0$, then $\lambda = 3$, so $a_2 = 0$ and $a_1 = 3a_3$,
• if $a_3 = 0$, $a_2 \neq 0$, then $\lambda = 2$, so $a_1 = 0$ and $a_0 = a_2$,
• if $a_3 = 0$, $a_2 = 0$, $a_0 \neq 0$, then $\lambda = 0$, so $a_1 = 0$,
• if $a_3 = 0$, $a_2 = 0$, $a_0 = 0$, we must have $a_1 \neq 0$ as $f$ is non-zero, so $\lambda = 1$.

As a result, we have 4 eigenvalues $\lambda \in \{0, 1, 2, 3\}$. (b) As seen in part (a), we already know the eigenvectors corresponding to the computed eigenvalues. Indeed, the following eigenvectors generate the eigenspace for each eigenvalue $\lambda$:

• for $\lambda = 3$, all vectors in the eigenspace look like $f(x) = 3a_3x + a_3x^3$, so $v_3 = 3x + x^3$ is a generating eigenvector,
• for $\lambda = 2$, all vectors in the eigenspace look like $f(x) = a_2 + a_2x^2$, so $v_2 = 1 + x^2$ is a generating eigenvector,
• for $\lambda = 0$, all vectors in the eigenspace look like $f(x) = a_0$, so $v_0 = 1$ is a generating eigenvector,
• for $\lambda = 1$, all vectors in the eigenspace look like $f(x) = a_1x$, so $v_1 = x$ is a generating eigenvector.

(c) Note that a basis of $V$ for which the matrix of $T$ would be diagonal must consist of eigenvectors of $T$. In this case, we know that eigenvectors for different eigenvalues are linearly independent, so the 4 eigenvectors $v_3, v_2, v_0, v_1$ above will be linearly independent and thus will form a basis for $V$. It’s easy to check that the matrix of $T$ in this basis will be:

$$
\begin{pmatrix}
3 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

Also note that since we found 4 eigenvalues, each has to have algebraic multiplicity 1 and thus for each eigenvalue, its algebraic multiplicity is the same as the geometric multiplicity, which tells us that $T$ is diagonalizable in the first place.

Remark that the first two parts could have been solved by using the matrix representation of $T$, finding the characteristic polynomial to figure out what the eigenvalues are and then solving to find a basis for the corresponding kernels. \(\square\)