1 (Reading).

2. (30 pts) [Ch. 3, Problem 5.2, Problem 5.4]

Solution. Problem 5.2 For the first matrix, expanding along the last column we have
\[ \begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 5 \\ 1 & -3 & 0 \end{vmatrix} = 5 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} = 25. \]

For the second matrix, expanding along the second row we have
\[ \begin{vmatrix} 4 & -6 & -4 & 4 \\ 2 & 1 & 0 & 0 \\ 0 & -3 & 1 & 3 \\ -2 & 2 & -3 & -5 \end{vmatrix} = 2 \cdot (-1)^{2+1} \cdot \begin{vmatrix} -6 & -4 & 4 \\ -3 & 1 & 3 \\ -2 & -3 & -5 \end{vmatrix} + (-1)^{1+2} \cdot \begin{vmatrix} 4 & -4 & 4 \\ 0 & 1 & 3 \\ -2 & -3 & -5 \end{vmatrix} + (-1)^{2+2} \cdot \begin{vmatrix} 4 & 4 \\ -2 & -5 \end{vmatrix} + 3 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 4 & -4 \\ -2 & -3 \end{vmatrix} = -32. \]

Problem 5.4 The inverses are
\[ \begin{pmatrix} -2 & 1 \\ \frac{2}{7} & -\frac{1}{7} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}, \]
respectively.

□

3. (30 pts) [Ch. 4, Problem 1.2]

Solution. For the first matrix, the characteristic polynomial is \( \lambda^2 - \lambda - 2 \). The eigenvalues are \(-1\) and \(2\). An eigenvector corresponding to \(-1\) is \((1, 1)\). An eigenvector corresponding to \(2\) is \((\frac{5}{2}, 1)\).

For the second matrix, the characteristic polynomial is \( \lambda^2 - 6\lambda - 9 = (\lambda - 3)^2 \). \(-3\) is an eigenvalue of algebraic multiplicity 2. The eigenspace has dimension 1,
with basis \((1,1)\).

For the third matrix, the characteristic polynomial is \(-\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2\). 1 is an eigenvalue of algebraic multiplicity 1 and \(-2\) is an eigenvalue of algebraic multiplicity 2. An eigenvector corresponding to 1 is \((1,-1,1)\). A basis for the eigenspace corresponding to -2 is \{\((-1,1,0),(-1,0,1)\)\}.

\(\square\)

4. (20 pts)[Ch. 3, Problem 5.6]

Solution. Vandermonde determinant revisited. Our goal is to prove the formula

\[
\begin{vmatrix}
1 & c_0 & c_0^2 & \cdots & c_0^n \\
1 & c_1 & c_1^2 & \cdots & c_1^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & c_n & c_n^2 & \cdots & c_n^n \\
\end{vmatrix} = \prod_{0 \leq j < k \leq n} (c_k - c_j)
\]

for the \((n \times 1) \times (n + 1)\) matrix. We will apply induction. To do this

a) Check that the formula holds for \(n = 1\) and \(n = 2\).

Plug in the dimensions and compute directly gives

\[
|1| = 1, \begin{vmatrix} 1 & c_0 \\ 1 & c_1 \end{vmatrix} = c_1 - c_0
\]

The left one is the product of zero terms, which is defined to be zero, and the right is the correct form for \(\prod_{0 \leq j < k \leq n} (c_k - c_j)\) when \(n = 2\).

b) Call the variable \(c_n\) in the last row \(x\), and show that the determinant is a polynomial of degree \(n\) in \(A\) with coefficients \(A_i\) depending only on \(c_0, ..., c_{n-1}\).

Perform the process of cofactor expansion on the bottom row. We get that the determinant is

\[A_0 + A_1 x + A_2 x^2 + ... + A_n x^n\]

where \(C_j\) is the determinant of the matrix obtained by deleting the bottom row and the \(i^{th}\) column. All these matrices have only terms among \(\{1, c_0, ..., c_{n-1}\}\) and thus has determinant depending only on those variables.

c) Show the polynomial as zeroes at \(x = c_0, c_1, ..., c_{n-1}\) so it can be represented as \(A_n(x - c_0) \cdot ... \cdot (x - c_{n-1})\) where \(A_n\) as above.

If we set \(x = c_0, ..., c_{n-1}\) then the matrix will have two identical rows. Then the determinant will vanish because the rows are linearly dependent. By properties of polynomials, it can be put in the form required by the question.

d) Assuming that the formula for the Vandermonde determinant is true for \(n - 1\), compute \(A_n\) and prove the formula for \(n\).

The term \(A_n\) is the determinant of the matrix obtained by deleting the \(n + 1^{th}\) column and row. That is a matrix of dimension \(n \times n\) that is also in the Vandermonde form. By induction hypothesis,

\[
A_n = \begin{vmatrix}
1 & c_0 & c_0^2 & \cdots & c_0^{n-1} \\
1 & c_1 & c_1^2 & \cdots & c_1^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & c_{n-1} & c_{n-1}^2 & \cdots & c_{n-1}^{n-1} \\
\end{vmatrix} = \prod_{0 \leq j < k \leq n-1} (c_k - c_j)
\]
multiplying $A_n$ by $(x - c_0) \cdot \ldots \cdot (x - c_{n-1})$ and setting $x = c_n$ hen gives the answer to the determinant for the $n$ dimension as $\prod_{0 \leq j < k \leq n}(c_k - c_j)$, which is what was desired to be shown.

5. (20 pts) Let $A$ be a $4 \times 4$ matrix be given by

$$A = \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}$$

Find the eigenvalues, and the characteristic polynomial. (Hint: A possible method is to first find the eigenvalues of $A + I$).

Solution. Let $v$ be a vector and $\lambda$ be a scalar. Then

$$(A + I)v = \lambda v \iff Av + v = \lambda v \iff Av = (\lambda - 1)v$$

Therefore the eigenvalues of $A$ are in one to one correspondence with the eigenvalues of $A - I$ by the map $\lambda \mapsto \lambda - 1$. The matrix $A + I$ is the matrix with every entry 1, and has rank 1, with image generated by $(1, 1, 1, 1)^T$. Therefore the only possible eigenvector corresponding to an non-zero eigenvalue is a multiple of $(1, 1, 1, 1)^T$. Computing $(A + I)(1, 1, 1, 1)^T = 4 \cdot (1, 1, 1, 1)^T$ gives that the only non-zero eigenvalue of $A + I$ is 4. On the other hand, the dimension of the $\ker(I + A)$ is 3, so 0 will be an eigenvalue of $A + I$ with geometric multiplicity 3. As a result, the algebraic multiplicity of 0 must be 3 as well, which forces the algebraic multiplicity of the eigenvalue 4 to be 1. As a result, $A$ has eigenvalues 3 and $-1$ with multiplicities 1 and 3 and thus the characteristic polynomial is given by $(x - 3)(x - 1)^3$.\qed