1. Prove the following special case of the Snake Lemma: given a commutative diagram of $A$-modules (dark arrows)

\[
\begin{array}{ccccccccc}
\ker \alpha & \rightarrow & \ker \beta & \rightarrow & \ker \gamma & \rightarrow & 0 \\
\downarrow f & & \downarrow g & & \downarrow h & & \partial \\
0 & \rightarrow & M_1 & \rightarrow & M_2 & \rightarrow & M_3 & \rightarrow & 0 \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & & & \\
0 & \rightarrow & N_1 & \rightarrow & N_2 & \rightarrow & N_3 & \rightarrow & 0 \\
\downarrow k & & \downarrow \gamma & & & & & & \\
& \rightarrow & \coker \alpha & \rightarrow & \coker \beta & \rightarrow & \coker \gamma & \rightarrow & 0
\end{array}
\]

with exact rows, show that there is an exact sequence

\[
0 \rightarrow \ker(\alpha) \rightarrow \ker(\beta) \rightarrow \ker(\gamma) \rightarrow \coker(\alpha) \rightarrow \coker(\beta) \rightarrow \coker(\gamma) \rightarrow 0.
\]

**Solution:**

First we define the maps

- $h \circ \alpha(\ker \alpha) = h(0) = 0$ hence $\beta \circ f(\ker \alpha) = 0$ which implies that $\ker \alpha$ maps to $\ker \beta$ under $f$. We define the first arrow to be $f|_{\ker \alpha}$.

- Same argument shows that $g|_{\ker \beta}$ maps $\ker \beta$ to $\ker \gamma$.

- Given an element $n \in N_1 / \text{Im } \alpha = \text{coker } \alpha$, we define its image to be $h(n)$ in $\text{coker } \beta$. We need to check that this is well defined, i.e. given to different representatives $n$ and $n'$ they both map to the same element in $\text{coker } \beta$. We have that $n - n' = \alpha(m)$ for some $m \in M_1$. Thus, $h(n) - h(n') = h(n - n') = h(\alpha(m)) = \beta(f(m)) \in \text{Im } \beta$.

- Same argument shows that we have a map from $\text{coker } \beta$ to $\text{coker } \gamma$.

- Since $g$ is surjective, given $m_3 \in \ker \gamma \subset M_3$, $m_3 = g(m_2)$ for $m_2 \in M_2$. We have $k \circ \beta(m_2) = \gamma \circ g(m_2) = \gamma(m_3) = 0$. Therefore, $\beta(m_2)$ is in the kernel of $k$ and by exactness of the second row $\beta(m_2) = h(n_1)$ for a unique $n_1 \in N_1$. We define $\partial$ to be the map that sends $m_3$ to $n_1$. We need to show that this is well defined. The only choice we have is $m_2$, given another one $m'_2$, $m_2 - m'_2$ is in the kernel of $g$. Therefore $m_2 - m'_2 = f(m_1)$ for $m_1 \in M_1$. In this case $\beta(m_2) - \beta(m'_2) = h(\alpha(m_1))$, hence $h^{-1}$ of the difference is in the image of $\alpha$ and these two elements give use the same class in $\text{coker } \alpha$.

Now we have all the maps that we need, we show the sequence is exact.
• Exactness at ker $\alpha$: The sequence is exact at ker $\alpha$ because its restriction of an injective morphism to a subset.

• Exactness at ker $\beta$: $m_2 \in \ker \beta$ goes to zero in $\ker \gamma$ if and only if $g(m_2) = 0$. By exactness of first row, this is equivalent to $m_2 = f(m_1)$. But $h \circ \alpha(m_1) = \beta(m_2) = 0$. By injectivity of the $h$, $\alpha(m_1) = 0$ and $m_1$ is in the ker $\alpha$. On the other hand given $m_1$ in ker $\alpha$ we have $g \circ f(m_1) = 0$, hence elements in ker $\beta$ that go to zero are exactly elements that are in the image of ker $\alpha$.

• Exactness at ker $\gamma$: Consider $m_3 \in \ker \gamma$ with $\partial(m_3) = 0$. By definition we have $h^{-1} \circ \beta \circ g^{-1}(m_3)$ is zero in coker $\alpha$, i.e. it is in the $\text{Im} \alpha$ and is equal to $\alpha(m_1)$. We apply $h$ to it. We get that $h \circ \alpha(m_1) = \beta \circ g^{-1}(m_3)$. But we have $h \circ \alpha = \beta \circ f$, which implies $\beta \circ f(m_1) = \beta \circ g^{-1}(m_3)$. Thus, $f(m_1) - g^{-1}(m_3) \in \ker \beta$. Applying $g$ we get $g \circ f(m_1) - m_3 \in \text{Im}(\ker \beta)$. By exactness of first row we have $f(m_1) \in \text{ker} g$ which implies that $m_3$ is in the image of ker $\beta$. On the other hand any element of ker $\gamma$ coming from ker $\beta$ goes to zero by $h^{-1} \circ \beta \circ g^{-1}$.

• Exactness at coker $\alpha$: Given $n_1 + \text{Im} \alpha \in \text{coker} \alpha$ that maps to zero in coker $\beta$, we have $h(n_1) = \beta(m_2)$. We claim that $g(m_2)$ is in ker $\gamma$ and maps to $n_1$. Note that $\gamma \circ g(m_2) = k \circ \beta(m_2) = k \circ h(n_1) = 0$. By definition we have $\partial(g(m_2)) = h^{-1} \circ \beta(m_2) = n_1$. On the other hand any element of the form $\partial(m_3)$ is equal to $h^{-1} \circ \beta \circ g^{-1}(m_3)$ hence it goes to $\beta \circ g^{-1}(m_3)$ under $h$ and is zero in coker $\beta$.

• Exactness at coker $\beta$: Given $n_2 + \text{Im} \beta \in \text{coker} \beta$ which goes to zero in coker $\gamma$, we have $k(n_2) = \gamma(m_3)$. Let $m_2$ be a preimage of $m_3$. We have $k \circ \beta(m_2) = \gamma \circ g(m_2) = k(n_2)$. Hence $k(-\beta(m_2)+n_2) = 0$. This means that we can replace $n_2$ with another representative in coker $\beta$ that actually goes to zero in $N_3$. By exactness we know such elements are in the image of $h$. On the other hand we have that $k \circ h = 0$.

• Exactness at coker $\gamma$: Follows from surjectivity of $k$.

2. Prove the following form of the 5-Lemma: Given a commutative diagram of $A$-modules

\[
\begin{array}{cccccc}
M_0 & \xrightarrow{f_0} & M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \xrightarrow{f_3} & M_4 \\
N_0 & \xrightarrow{g_0} & N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 & \xrightarrow{g_3} & N_4 \\
\end{array}
\]

with exact rows such that $\alpha$ and $\gamma$ are isomorphisms, $\omega$ is surjective, and $\epsilon$ is injective, then $\beta$ is an isomorphism.

**Solution:**
Since the diagram is commutative ($\gamma \circ f_2 = g_2 \circ \beta$ and $\beta \circ f_1 = g_1 \circ \alpha$), we get the following commutative diagram.

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Since $\gamma$ is isomorphisms, its restrictions is injective. We show that the restrictions is also surjective. Given $n_3 \in \text{Im} \ g_2$, there is unique element $m_3 \in M_3$ that gets mapped to it by $\gamma$. Let $m_4 = f_3(m_3)$ be its image in $M_4$. We have

$$\epsilon(m_4) = \epsilon \circ f_3(m_3) = g_3 \circ \gamma(m_3) = g_3(n_3) = 0.$$ 

The last equality follows from the fact that $g_3 \circ g_2 = 0$. Since $\epsilon$ is injective, we have $m_4 = 0$, i.e. $m_3 \in \ker f_3 = \text{Im} f_2$. It follows that $\gamma|_{\ker f_2}$ is an isomorphism.

Note that $\tilde{\alpha}$ is well defined since the diagram is commutative. $\tilde{\alpha}$ is clearly surjective since $\alpha$ is surjective. We show that $\tilde{\alpha}$ is injective. Let $m_1 \in M_1$ such that $\alpha(m_1) \in \ker g_1$, we need to show that $m_1 \in \ker f_1$. We have

$$\alpha(m_1) \in \ker g_1 = \text{Im} g_0 \quad \alpha(m_1) = g_0(n_0) = g_0 \circ \omega(m_0) \quad m_0 \in M_0$$

The last equality follows from surjectivity of $\omega$. Note that $g_0 \circ \omega = \alpha \circ f_0$ and we get $\alpha \circ f_0(m_0) = \alpha(m_1)$. Since $\alpha$ is and isomorphism we have $m_1 = f_0(m_0) \in \text{Im} f_0 = \ker f_1$.

We use problem (1) to finish the proof. By problem (1) we have the following exact sequence:

$$0 \to \ker(\tilde{\alpha}) \to \ker(\beta) \to \ker(\gamma|_{\text{Im} f_2}) \to \text{coker}(\tilde{\alpha}) \to \text{coker}(\beta) \to \text{coker}(\gamma|_{\text{Im} f_2}) \to 0.$$ 

We have shown that $\tilde{\alpha}$ and $\gamma|_{\text{Im} f_2}$ are isomorphisms, therefore we have:

$$0 \to 0 \to \ker(\beta) \to 0 \to 0 \to \text{coker}(\beta) \to 0 \to 0.$$ 

This implies that kernel and cokernel of $\beta$ are zero, i.e. $\beta$ is an isomorphism.

3. (Origin of the name Tor, as in torsion) Let $a$ be a nonzerodivisor in a ring $A$. Show that

$$\text{Tor}_1^A(A/a, M) = \{m \in M | m \not\in aM \}.$$ 

Note that this determines $\text{Tor}_1^A(M, N)$ for all finitely generated abelian groups $M$ and $N$. Also, the higher Tors vanish in this case.

Solution:

There is a simple resolution of $A/a$ by free (hence projective) modules:

$$0 \to A \xrightarrow{x} A \to A/a \to 0.$$ 

By definition $\text{Tor}_i(A/a, M)$’s are homologies of the following complex:

$$0 \to A \otimes M \xrightarrow{x} A \otimes M \to 0.$$ 

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Note that the first homology is the kernel of multiplication by $a$ in $M$, which is exactly the set $\{m \in M | am = 0\}$.

4. Suppose

$$0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} 0$$

is a complex of finite length $A$-modules. Prove that

$$\sum (-1)^i \ell(M_i) = \sum (-1)^i \ell(H_i(M_*))$$

Your argument should apply to any invariant that (like length) is additive in (short) exact sequences.

Solution:
We construct two sets of short exact sequences.

$$0 \rightarrow \ker f_i \rightarrow M_i \rightarrow \im f_i \rightarrow 0$$

$$0 \rightarrow \im f_{i-1} \rightarrow \ker f_i \rightarrow H_i(M_*) \rightarrow 0$$

First one follows from definition of kernel and image and the second one follows from definition of homology. Since length is additive, we get two sets of equalities

$$\ell(M_i) = \ell(\ker f_i) + \ell(\im f_i)$$

$$\ell(H_i(M_*)) = \ell(\ker f_i) - \ell(\im f_{i-1})$$

We take alternating sums of these two equalities,

$$\sum (-1)^i \ell(M_i) = \sum (-1)^i (\ell(\ker f_i) + \ell(\im f_i)) = \sum (-1)^i \ell(\ker f_i) + \sum (-1)^i \ell(\im f_i)$$

$$\sum (-1)^i \ell(H_i(M_*)) = \sum (-1)^i (\ell(\ker f_i) - \ell(\im f_{i-1})) = \sum (-1)^i \ell(\ker f_i) + \sum (-1)^{i-1} \ell(\im f_{i-1})$$

Right hand sides are equal, therefore the left hand side are equal too.