You can use any of the references you’ve been using for the course to work on these problems, but not if you literally find the solution to a problem. Also, no collaboration for this week.

(1) If $M$ is an $A$ module and $N$ and $N'$ are submodules of $M$, and $S \subset A$ is a multiplicative set, show that $N_S \cap N'_S = (N \cap N')_S$. Here $N_S$ denotes the localization $S^{-1}N$.

(2) Let $k$ be a field and let $R = k[x_1, \ldots, x_n]$. Show that for any prime ideal $P \subset R$ we have $ht P + coht P = n$. (i.e. There is a chain of prime ideals of length $n$ in $R$ which includes $P$ as a member.)

(3) Given $A$ modules $M$ and $N$, consider the following set of short exact sequences of $A$-modules:

$E^1(N, M) = \{ 0 \to M \to E \to N \to 0 \} / \cong$

where two short exact sequences are considered equivalent if there is a diagram

\[
\begin{array}{ccc}
0 & \to & M \\
\downarrow{id} & & \downarrow{\beta} \\
0 & \to & E \\
\downarrow{\beta} & & \downarrow{id} \\
& & 0
\end{array}
\]

(The middle vertical arrow is forced to be an isomorphism by the 5-Lemma.) This is called the set of extensions of $N$ by $M$. Show that there is a canonical bijection between $E^1(N, M)$ and $\text{Ext}^1_A(N, M)$ as follows. Choose a surjection from a projective module $P \to N$, and let $K$ be the kernel. Show that we can construct a commutative diagram

\[
\begin{array}{ccc}
0 & \to & K \\
\downarrow{v} & & \downarrow{id} \\
0 & \to & M \\
\downarrow{v} & & \downarrow{\beta} \\
0 & \to & E \\
\end{array}
\]

Now apply the functor $\text{Hom}_A(\cdot, M)$ to the top row of this diagram and use the fundamental property of $\text{Ext}$ to get a map from $\text{Hom}(K, M)$ to $\text{Ext}^1(N, M)$. The desired class is then the image of $v$ under this map. Now show that this map from $E^3$ to $\text{Ext}^1$ is well defined and a bijection. (See page 831 in Lang’s Algebra Exercise XX.27 for further hints here.)

(Note: another approach is to directly define the map by applying the functor $\text{Hom}(N, \cdot)$ to the bottom row of the diagram, and defining the desired class in $\text{Ext}^1(N, M)$ as the image of the identity element in $\text{Hom}(N, N)$ in the resulting long exact sequence. You can use whichever method you prefer – they are the same map but you don’t need to check this.)

Finally, the result of this problem implies that there is a natural structure of $A$-module on the set $E^1(N, M)$. For extra credit, describe explicitly how to “add” two extensions or multiply an extension by an element of $A$. 