(1) (a) Show that for any ring $A$, any $A$ module $M$ and any ideal $I \subset A$, we have

$$A/I \otimes_A M \cong M/IM.$$ 

(b) Show that for ideals $I$ and $J$, we have $A/I \otimes A/J \cong A/(I + J)$.

(2) Suppose that

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of $A$ modules. Show that the sequence is split if and only if the induced map $\text{Hom}_A(M'', M) \to \text{Hom}_A(M'', M'')$ induced by the right hand map in the sequence is surjective. Use this to show that if $M''$ is finitely presented, then the sequence is split if and only if the localized sequences

$$0 \to M'_m \to M_m \to M''_m \to 0$$

are split for all maximal ideals $m \subset A$.

(3) (a) Show that if $M$ is an $A$ module and $B$ is an $A$-algebra, then $B \otimes_A M$ is naturally a $B$-module.

(b) Show that if $B$ and $C$ are $A$-algebras, then $B \otimes_A C$ has a natural ring structure which makes it both a $B$-algebra and a $C$-algebra.

(c) Show that in the above situation, $B \otimes_A C$ satisfies the following universal property. Given any ring $R$ and ring homomorphisms $f : B \to R$ and $g : C \to R$ such that the induced morphisms from $A$ to $R$ agree, there exists a unique morphism from $B \otimes_A C$ to $R$ making the diagram below commute:

```
\begin{tikzcd}
A \arrow[dr] & B \arrow[l] \arrow[d] \arrow[r] & R. \arrow[dl] \\
& B \otimes_A C \arrow[r] & \\
& C \arrow[ur] &
\end{tikzcd}
```

(4) Show that the tensor product of two flat $A$ modules is flat.

(5) Show that if $M$ is a flat $A$ module, and if $a$ is an element of $A$ which is not a zero divisor, then $a$ is not a zero divisor in $M$, i.e. if $am = 0$ for some $m \in M$ we must have $m = 0$. 