(1) Suppose that a ring $A$ has the property that for every prime ideal $p$ of $A$, the local ring $A_p$ contains no nonzero nilpotents. Show that $A$ contains no nonzero nilpotents. If each $A_p$ is an integral domain, is $A$ necessarily an integral domain?

(2) For a ring $A$ and a multiplicative subset $S$, show that an $S^{-1}A$ module is the same thing as an $A$ module on which the elements of $S$ act as automorphisms. In particular, the natural map from $M$ to $S^{-1}M$ is an isomorphism if and only if every element of $S$ acts invertibly on $M$.

(3) We say a ring $R$ is $\mathbb{Z}$-graded if we have an infinite direct sum decomposition

$$R = \cdots \oplus R_{-2} \oplus R_{-1} \oplus R_0 \oplus R_1 \oplus \cdots$$

and $R_i \cdot R_j \subset R_{i+j}$. (A common situation is when all the negative terms vanish, in which case we would say $R$ is a graded ring.) An element of $R$ is called homogeneous if it lies in one of the $R_i$. Note that by definition, every element $f$ in $R$ can be written uniquely in the form $f = \sum f_i$ as a finite sum of homogeneous elements which are called the homogeneous components of $f$. In a graded or $\mathbb{Z}$-graded ring, an ideal is called homogeneous if it is generated by homogeneous elements.

(a) Show that an ideal $I$ is homogeneous if and only if for every $f$ in $I$, all the homogeneous components of $f$ are in $I$.

(b) Show that the radical of a homogeneous ideal is homogeneous.

(b) Show that a homogeneous ideal $I$ is prime if and only if for all homogeneous elements $f, g$ in $R$, we have $fg \in I$ if and only if $f \in I$ or $g \in I$.

(4) Let $R$ be a $\mathbb{Z}$-graded ring, and $f$ is an element of $R_1$.

(a) Show that $R_f = R[f^{-1}]$ is naturally $\mathbb{Z}$-graded. (What must the degree of $\frac{1}{f}$ be?) Let $A$ be the degree zero graded piece of $R_f$.

(b) Show that $R_f \cong A[x, x^{-1}]$, the ring of Laurent polynomials with coefficients in $A$. (Here $x$ is just a new formal variable.)

(c) Show that $A \cong R/(f - 1)$.