Let $A$ be a ring, and let $I$ be an ideal contained in $\text{nil}(A)$. (i.e. Every element of $I$ is nilpotent.) Show that if $a \in A$ maps to a unit in $A/I$, then $a$ is a unit in $A$. Give an example to show that the hypothesis $I \subset \text{nil}(A)$ is necessary.

(2) Show that for any ideal $I$ in a ring, there exists a prime minimal among those containing $I$. (Hint: Show that the intersection of a decreasing chain of primes is prime.)

(3) Show that if an ideal is contained in a finite union of prime ideals, then it must be contained in one of them.

(4) Show that every prime ideal in a product ring $R \times R'$ is of the form $p \times R'$ for $p$ a prime in $R$ or $R \times p'$ for $p'$ a prime ideal in $R'$.

(5) A multiplicative subset $S \subset R$ is called saturated if $xy \in X$ implies that both $x$ and $y$ are in $S$. Show that:

(a) $S$ is saturated if and only if the complement of $S$ is a union of prime ideals.
(b) If $S$ is any multiplicatively closed subset of $R$, there is a unique smallest saturated multiplicatively closed subset $\overline{S}$ containing $S$, and that $\overline{S}$ is the complement of the union of the prime ideals not meeting $S$.
(c) Show that $\overline{S}$ is the largest multiplicative subset containing $S$ such that the natural map from $S^{-1}R$ to $\overline{S}^{-1}R$ is an isomorphism.

(6) (a) Suppose $I$ is an ideal whose radical is finitely generated – that is, $\sqrt{I} = \langle a_1, \ldots, a_r \rangle$ for some collection of elements $a_1, \ldots, a_r$ in $R$. Show that $(\sqrt{I})^n \subset I$ for sufficiently large $n$.

(b) Suppose $I$ is an ideal and $p$ is a finitely generated prime ideal. Show that $\sqrt{I} = p$ if and only if there exists an $n > 0$ such that $p^n \subset I \subset p$. 