EXERCISE SET # 1

(1) Show that the following functions are primitive recursive (on $A = \{\sigma_1, \ldots, \sigma_k\}$):

(a) $\text{FIRST}(x) = \begin{cases} \theta, & \text{if } x = \theta, \\ \sigma_j, & \text{if } x = \sigma_j y. \end{cases}$

(b) $\text{LAST}(x) = \begin{cases} \theta, & \text{if } x = \theta, \\ \sigma_j, & \text{if } x = y\sigma_j. \end{cases}$

(c) $\text{RESTR}(x, y) = x[|y|]$

(d) $\text{LENGTH}(x) = \sigma_1^{|x|}$

(2) For each $f : A^* \rightarrow A^*$ define its $n$th iterate, $f^n : A^* \rightarrow A^*$ by:

$$f^0(x) = x,$$
$$f^{n+1}(x) = f(f^n(x)).$$

(Thus $f^1 = f.$) Show that if $f$ is primitive recursive, so is $f^*(w, x) = f^{|w|}(x)$.

(3) Prove that the following are primitive recursive (on $A = \{\sigma_1, \ldots, \sigma_k\}$).

(a) $\text{PALIN}(x) \iff x$ is a palindrome $\iff x = x^*$

(b) $U(x, z) = \begin{cases} \text{The word to the left of the leftmost occurrence of } x \text{ in } z, \text{ if } x \text{ occurs in } z, \\ x\sigma_1, \text{ otherwise}. \end{cases}$

(c) $V(x, z) = \begin{cases} \text{The word to the right of the leftmost occurrence of } x \text{ in } z, \text{ if } x \text{ occurs in } z, \\ \theta, \text{ otherwise}. \end{cases}$
REP\((x, y, z) = \begin{cases} 
\text{The result of replacing the leftmost occurrence of } x \text{ in } z \text{ by } y, \text{ if } x \text{ occurs in } z, \\
z\sigma_1 y, \text{ otherwise.} 
\end{cases} \)

\(F(x) = \) the maximal (longest) solution \(w\) of the equation \(x = wyw^*\).

(4) A \textit{pairing function} on \(\mathbb{N}\) is a bijection \(\langle , \rangle: \mathbb{N}^2 \to \mathbb{N}\) such that \(\langle x, y \rangle < \langle x, y + 1 \rangle, \langle x, y \rangle < \langle x + 1, y \rangle\), i.e., \(\langle , \rangle\) is increasing in each argument. Denote by \(\pi_1, \pi_2\) the two projection functions, defined by \(x = \langle \pi_1(x), \pi_2(x) \rangle\). Note that \(x, y \leq \langle x, y \rangle\), thus \(\pi_1(x), \pi_2(x) \leq x\).

Example. \(\langle x, y \rangle = 2^x \cdot (2y+1) - 1\) is a primitive recursive pairing function.

Note that the two projection functions of any primitive recursive pairing function are also primitive recursive, since
\[
\pi_1(x) = \mu y \leq x \exists z \leq x(\langle y, z \rangle = x) \\
\pi_2(x) = \mu z \leq x \exists y \leq x(\langle y, z \rangle = x).
\]

Consider now the Cantor enumeration of \(\mathbb{N}^2\):

\[
\begin{array}{cccc}
0,0 & 0,1 & 0,2 & 0,3 & \cdots \\
1,0 & 1,1 & 1,2 & 1,3 & \cdots \\
2,0 & 2,1 & 2,2 & 2,3 & \cdots \\
3,0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots 
\end{array}
\]
given by: \((0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0), (0, 3), (1, 2), (2, 1), (3, 0), \ldots\)
(a) Let \( \langle x, y \rangle \) be the place of \( (x, y) \) in this enumeration (e.g. \( \langle 0, 0 \rangle = 0, \langle 0, 1 \rangle = 1, \langle 1, 0 \rangle = 2, \ldots \)). Show that
\[
\langle x, y \rangle = \frac{(x + y)(x + y + 1)}{2} + x.
\]
(b) Given a primitive recursive pairing function \( \langle \cdot, \cdot \rangle \), recursively construct for each \( n \geq 2 \) a primitive recursive bijection of \( \mathbb{N}^n \) with \( \mathbb{N}, \langle \cdots \rangle_n \) as follows:
\[
\langle x_1, x_2 \rangle_2 = \langle x_1, x_2 \rangle \\
\langle x_1, x_2, \cdots, x_n, x_{n+1} \rangle_{n+1} = \langle \langle x_1, x_2 \rangle, x_3, \cdots, x_{n+1} \rangle_n
\]
Show that there is a primitive recursive function \( \pi(i, n, x) \) such that
\[
\langle x_1, \cdots, x_n \rangle = y \Rightarrow \pi(i, n, y) = x_i
\]
for all \( n \geq 2, 1 \leq i \leq n. \)
(Hint: Use recursion on \( n \geq 2. \))

(5) Let \( A \) be an alphabet and let \( P \) be a relation on \( A^* \). Let
\[
Q(z, x_1, \ldots, x_n) \iff \exists y(|y| \leq |z| \ \& \ P(y, x_1, \ldots, x_n)).
\]
If \( P \) is primitive recursive on \( A \), then so is \( Q \).