Math 108A Problem Set 7, Part II Sample Solutions

**Problem 1:** Let \( M_{m,n} \) denote the set of all real-valued \( m \times n \) matrices. For any \( C \in M_{m,n} \), define \( M = \sum_{i=1}^{n} ||Ce_i|| \). Then for any \( x \in \mathbb{R}^n \) with \( ||x|| = 1 \), we have

\[
||Cx|| = \left| C \left( \sum_{i=1}^{n} x_i e_i \right) \right| = \left| \sum_{i=1}^{n} x_i (Ce_i) \right| \leq \sum_{i=1}^{n} |x_i||Ce_i| \leq \sum_{i=1}^{n} ||Ce_i|| = M,
\]

since each \( |x_i| \leq 1 \). In particular, for any \( A, B \in M_{m,n} \), the set \( R_{A,B} = \{||(A - B)x|| : x \in \mathbb{R}^n, ||x|| = 1\} \) is bounded above by \( M_{A-B} \). Since we also have that \( 0 \leq ||(A - B)e_1|| \in R_{A,B} \), it follows that \( 0 \leq d(A, B) \leq M_{A-B} < \infty \), and so \( d \) indeed maps \( M_{m,n} \times M_{m,n} \) into \([0, \infty)\).

Consider any \( A, B, C \in M_{m,n} \). Since \( ||(A - A)x|| = 0 \) for all \( x \in \mathbb{R}^n \) with \( ||x|| = 1 \), it follows that 0 is an upper bound for \( R_{A,A} \), and so \( d(A, A) = 0 \). Suppose that \( A \neq B \). Letting \( A = [a_{ij}] \) and \( B = [b_{ij}] \), it follows that for some \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), we have that \( a_{ij} \neq b_{ij} \). Hence the \((i, j)\)-th entry of \( A - B \) is \( a_{ij} - b_{ij} \neq 0 \). Therefore,

\[
||(A - B)e_j|| = \left| \left( \sum_{k=1}^{n} (a_{ik} - b_{ik}) (e_j)_k \right) \right| = |a_{ij} - b_{ij}| > 0.
\]

Since \( ||e_j|| = 1 \), it follows that \( d(A, B) \geq |a_{ij} - b_{ij}| > 0 \).

For any \( x \in \mathbb{R}^n \) with \( ||x|| = 1 \), we have that \( ||(A - B)x|| = ||(A - B)x|| = ||(B - A)x|| \). This shows that \( R_{A,B} = R_{B,A} \), and so it follows that \( d(A, B) = d(B, A) \).

For any \( x \in \mathbb{R}^n \) with \( ||x|| = 1 \), we have that

\[
||(A - B)x|| = ||Ax - Bx|| = ||Ax - Cx + Cx - Bx|| = ||(A - C)x + (C - B)x||
\]

\[
\leq ||(A - C)x|| + ||(C - B)x|| \leq d(A, C) + d(C, B).
\]

Hence \( d(A, C) + d(C, B) \) is an upper bound for \( R_{A,B} \). It follows that \( d(A, B) \leq d(A, C) + d(C, B) \). Therefore, \( d \) defines a metric on \( M_{m,n} \).

**Problem 2:** Since the functions \( (x, y) \mapsto x \) and \( (x, y) \mapsto y \) are continuous on \( \mathbb{R}^2 \) and \( x \mapsto \sqrt{x} \) is continuous on \([0, \infty)\), we can repeatedly apply Theorems 4.7 and 4.9 of Rudin to see that the function \( (x, y) \mapsto \frac{xy}{\sqrt{x^2 + y^2}} \) is continuous on \( \mathbb{R}^2 \setminus \{0\} \) (since if \((x,y) \neq 0\), then \( \sqrt{x^2 + y^2} \geq \max \{|x|, |y|\} > 0 \)).

So for any \((x, y) \neq 0\) and any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \((x', y') \neq 0\) and satisfies \( ||(x, y) - (x', y')|| < \delta \), then \( \left| \frac{xy}{\sqrt{x^2 + y^2}} - \frac{x'y'}{\sqrt{x'^2 + y'^2}} \right| < \varepsilon \). If we let \( \delta' = \min \{ \delta, ||(x, y)|| \} > 0 \), then \( ||(x, y) - (x', y')|| < \delta' \) implies that \((x', y') \neq 0\), and so \( |G(x, y) - G(x', y')| = \left| \frac{xy}{\sqrt{x^2 + y^2}} - \frac{x'y'}{\sqrt{x'^2 + y'^2}} \right| < \varepsilon \). This proves that \( G \) is continuous on \( \mathbb{R}^2 \setminus \{0\} \).

Given any \( \varepsilon > 0 \), let \( \delta = \varepsilon \). Then if \( 0 < ||(x, y) - 0|| = ||(x, y)|| < \delta \), we have

\[
|G(x, y) - G(0)| = \left| \frac{xy}{\sqrt{x^2 + y^2}} - 0 \right| = \frac{|x||y|}{\sqrt{x^2 + y^2}} \leq \frac{||(x, y)||^2}{||(x, y)||} = ||(x, y)|| < \delta = \varepsilon,
\]

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since $|x|, |y| \leq \|(x, y)\| = \sqrt{x^2 + y^2}$. It follows that $G$ is continuous at 0 as well, and so $G$ is continuous on all of $\mathbb{R}^2$.

Fix a $y_0 \in \mathbb{R}$. As with continuity, we can repeatedly apply Theorems 5.3 and 5.5 of Rudin to see that the function $x \mapsto \frac{x y_0}{\sqrt{x^2 + y_0^2}}$ is differentiable on all of $\mathbb{R}$, with the exception $(x, y_0) = 0$. Using the same argument as in the first paragraph, it follows that $G'_x$ exists on $\mathbb{R}^2 \setminus \{0\}$. By the same argument, $G'_y$ exists on $\mathbb{R}^2 \setminus \{0\}$ as well.

For the point 0, we compute

$$
\lim_{t \to 0} \frac{G(0 + (t, 0)) - G(0)}{t} = \lim_{t \to 0} \frac{G'(0) - 0}{t} = \lim_{t \to 0} \frac{t(0)}{\sqrt{t^2 + 0^2}} = \lim_{t \to 0} 0 = 0.
$$

Therefore, $G'_x(0) = 0$. By a similar calculation, we also see that $G'_y(0) = 0$. Hence both of the partial derivatives of $G$ exist on all of $\mathbb{R}^2$.

Suppose, for sake of contradiction, that $G$ were differentiable in the sense of Fréchet. In particular, the linear transformation $G'(0) : \mathbb{R}^2 \to \mathbb{R}$ exists. By Theorem 9.17 of Rudin, it follows that the matrix for $G'(0)$ is $[0 \ 0]$, since $G'_x(0) = G'_y(0) = 0$. Since $\lim_{h \to 0} \frac{|G(0 + h) - G(0) - G'(0)h|}{|h|} = 0$ by definition, it follows that there exists a $\delta > 0$ such that if $0 < |h| < \delta$, then $\frac{|G(0 + h) - G(0) - G'(0)h|}{|h|} < \frac{1}{2}$. Let $h_0 = (\frac{\delta}{2}, \frac{\delta}{2})$. Then $0 < |h_0| = \sqrt{(\frac{\delta}{2})^2 + (\frac{\delta}{2})^2} = \frac{\delta}{\sqrt{2}} < \delta$. However,

$$
\frac{|G(0 + h_0) - G(0) - G'(0)h_0|}{|h_0|} = \frac{|G(h_0) - 0 - [0 \ 0]h_0|}{|h_0|} = \frac{|(\frac{\delta}{2})^2 + (\frac{\delta}{2})^2|}{\frac{\delta}{\sqrt{2}}} = \frac{(\frac{\delta}{\sqrt{2}})(\frac{\delta}{\sqrt{2}})}{\frac{\delta}{\sqrt{2}}} = \frac{1}{2},
$$

a contradiction. Hence $G$ is not Fréchet differentiable.

**Problem 3:** Consult the proof of Theorem 9.15 of Rudin, which solves this problem for the case $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, and $Z = \mathbb{R}^k$. It can be verified that no special properties of Euclidean spaces were used (in particular, because Fréchet derivatives are bounded by definition), and all operations with the norm are valid in general normed spaces. Therefore, by replacing the $\mathbb{R}^n$ norm with $\|\cdot\|_X$, replacing $\|\cdot\|_\mathbb{R}^m$ with $\|\cdot\|_Y$, and replacing $\|\cdot\|_\mathbb{R}^k$ with $\|\cdot\|_Z$, the same proof will work.