Lemma 1: Suppose that $X$ is a metric space, $E \subseteq X$, and $\{f_n\}$ is an equicontinuous sequence of complex-valued functions defined on $E$ such that $f_n \to f$ on $E$. Then $f$ is uniformly continuous on $E$.

Proof: Consider any $\varepsilon > 0$. Since $\{f_n\}$ is equicontinuous, choose $\delta > 0$ such that for all $x, y \in E$ with $d(x, y) < \delta$, we have $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$ for any $n \in \mathbb{N}$. Consider any $x, y \in E$ with $d(x, y) < \delta$. Since $f_n \to f$ on $E$, let $N_1$ and $N_2$ be such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $n \geq N_1$ and $|f_n(y) - f(y)| < \frac{\varepsilon}{3}$ for all $n \geq N_2$. Then, letting $N = \max \{N_1, N_2\}$, we have that

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \quad \blacksquare$$

7.16. Solution 1: Consider any $\varepsilon > 0$. By equicontinuity of $\{f_n\}$, let $\delta_1 > 0$ be such that for any $x, y \in K$ with $d(x, y) < \delta_1$, we have $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$ for any $n \in \mathbb{N}$. Since $K$ is compact, the open cover $\{B_{\delta_1}(x) : x \in K\}$ of $K$ has a finite subcover: say $K \subseteq \bigcup_{i=1}^{M} B_{\delta_1}(x_i)$. Since for each $1 \leq i \leq M$, the sequence $\{f_n(x_i)\}$ converges, it is Cauchy, and so let $N_i$ be such that $|f_n(x_i) - f_m(x_i)| < \frac{\varepsilon}{3}$ for all $n, m \geq N_i$. Define $N = \max \{N_i : 1 \leq i \leq M\}$.

We claim that for all $x \in K$ and all $n, m \geq N$, we have that $|f_n(x) - f_m(x)| < \varepsilon$. Given any $x \in K$, since $K \subseteq \bigcup_{i=1}^{M} B_{\delta_1}(x_i)$, choose $1 \leq i \leq M$ such that $d(x, x_i) < \delta_1$. Then

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_m(x_i)| + |f_m(x_i) - f_m(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Therefore, $\{f_n\}$ is uniformly Cauchy on $K$, and so by Theorem 7.8 of Rudin, $\{f_n\}$ converges uniformly on $K$.

Solution 2: Since $\{f_n\}$ converges pointwise on $K$, let $f : K \to \mathbb{C}$ be the function so that $f_n \to f$. Since $\{f_n\}$ is equicontinuous on $K$, it follows from Lemma 1 that $f$ is uniformly continuous on $K$.

Consider any $\varepsilon > 0$. Let $\delta_1 > 0$ be as in Solution 1. By uniform continuity of $f$, let $\delta_2 > 0$ be such that for any $x, y \in K$ with $d(x, y) < \delta_2$, we have $|f(x) - f(y)| < \frac{\varepsilon}{3}$. Let $\delta = \min \{\delta_1, \delta_2\} > 0$. As in Solution 1, choose a finite set $\{y_1, \ldots, y_{M'}\}$ such that $K \subseteq \bigcup_{i=1}^{M'} B_\delta(y_i)$. Since $f_n \to f$, for each $1 \leq i \leq M'$, let $N'_i$ be such that $|f_n(y_i) - f(y_i)| < \frac{\varepsilon}{3}$ for all $n \geq N'_i$, and let $N' = \max \{N'_i : 1 \leq i \leq M'\}$.

We claim that for all $x \in K$ and all $n \geq N'$, we have that $|f_n(x) - f(x)| < \varepsilon$. Given any $x \in K$, since $K \subseteq \bigcup_{i=1}^{M'} B_\delta(y_i)$, choose $1 \leq i \leq M'$ such that $d(x, y_i) < \delta \leq \delta_1, \delta_2$. Then

$$|f_n(x) - f(x)| \leq |f_n(x) - f_n(y_i)| + |f_n(y_i) - f(y_i)| + |f(y_i) - f(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Therefore, $\{f_n\}$ converges uniformly on $K$.

7.25. For any $n \geq 1$ and $0 \leq i \leq n$, let $x_{n,i} = \frac{i}{n}$ (so each $x_{n,i} \in [0, 1]$). For a fixed $n \geq 1$, we will define $y_{n,i}$ for $0 \leq i \leq n$ by induction on $i$: first $y_{n,0} = c$. Then $y_{n,i+1} = y_{n,i} + \frac{1}{n} \phi(x_{n,i}, y_{n,i})$. Then define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(t) = y_{n,i} + \phi(x_{n,i}, y_{n,i})(t - x_{n,i}) \quad \text{if} \ t \in [x_{n,i}, x_{n,i+1}].$$
Note that if \( t = x_{n,i} \) with \( 0 < i < n \), then there are two definitions for \( f_n(t) \) that agree, and so \( f_n \) is well-defined on \([0, 1]\). Since \( f_n \) is continuous (in fact, linear) on each interval \([x_{n,i}, x_{n,i+1}]\), it follows that \( f_n(t) = f_n(t-) = f_n(t+) \) for all \( t \in (0, 1) \), and so \( f_n \) is continuous on all of \([0, 1]\). We also have that \( f_n(x_{n,i}) = y_{n,i} \) for all \( i \leq n \), and in particular, \( f_n(0) = f_n(x_{0,0}) = y_{0,0} = c \). Hence if \( t \in (x_{n,i}, x_{n,i+1}) \), then \( f'_n(t) = \phi(x_{n,i}, y_{n,i}) = \phi(x_{n,i}, f_n(x_{n,i})) \).

For any \( n \geq 1 \), define \( \Delta_n : [0, 1] \to \mathbb{R} \) by

\[
\Delta_n(t) = \begin{cases} 
0 & \text{if } t = x_{n,i} \text{ for some } i \leq n \\
 f'_n(t) - \phi(t, f_n(t)) & \text{otherwise.}
\end{cases}
\]

Since \( f'_n \) is continuous (in fact, constant) on each interval \((x_{n,i}, x_{n,i+1})\) and \( t \mapsto \phi(t, f_n(t)) \) is continuous on all of \([0, 1]\) (since \( \phi \) and \( f_n \) are continuous), it follows that \( \Delta_n \) is continuous on each interval \((x_{n,i}, x_{n,i+1})\). So the discontinuities of \( \Delta_n \) are among \( \{x_{0,0}, x_{1,1}, \ldots, x_{n,n}\} \), implying there are finitely many, and hence \( \Delta_n \) is Riemann integrable on \([0, 1]\) by Theorem 6.10 of Rudin. Since \( \phi(t, f_n(t)) \) is continuous, the sum \( \phi(t, f_n(t)) + \Delta_n(t) \) is also Riemann integrable.

For any \( i < n \), and \( x, y \) with \( x_{n,i} \leq x \leq y \leq x_{n,i+1} \), we have that \( \int_x^y \phi(t, f_n(t)) + \Delta_n(t) \, dt = \int_x^y \phi(t, f_n(t)) + f'_n(t) - \phi(t, f_n(t)) \, dt = \int_x^y f'_n(t) \, dt = f_n(y) - f_n(x) \) by the fundamental theorem of calculus, since \( f_n \) is differentiable when restricted to only the interval \([x, y]\), and \( f'_n \) is continuous and hence integrable on \((x, y)\). In particular, this implies that for any \( x \in [0, 1] \), letting \( j = \max \{i \leq n : x_{n,i} \leq x\} \),

\[
 c + \int_0^x (\phi(t, f_n(t)) + \Delta_n(t)) \, dt = c + \sum_{i=1}^j \int_{x_{n,i-1}}^{x_{n,i}} (\phi(t, f_n(t)) + \Delta_n(t)) \, dt + \int_{x_{n,j}}^x (\phi(t, f_n(t)) + \Delta_n(t)) \, dt
\]

\[
= c + \sum_{i=1}^j (f_n(x_{n,i-1}) - f_n(x_{n,i})) + f_n(x) - f_n(x_{n,j}) = f_n(x).
\]

By assumption, \( \phi \) is bounded, and so there exists \( 0 < M < \infty \) such that \( |\phi(x, y)| \leq M \) for all \((x, y) \in [0, 1] \times \mathbb{R}\).

(a) For any \( t \in [0, 1) \), choose \( i \leq n \) such that \( t \in [x_{n,i}, x_{n,i+1}) \). If \( f_n \) is differentiable at \( t \), then \( f'_n(t) = \lim_{h \to 0^+} \frac{f(t+h) - f(t)}{h} = \phi(x_{n,i}, f_n(x_{n,i})) \), and so \( |f'_n(t)| = |\phi(x_{n,i}, f_n(x_{n,i}))| \leq M \). Similarly, we have that \( |f'_n(1)| = |\lim_{h \to 0^-} \frac{f(t+h) - f(t)}{h}| = |\phi(x_{n,n-1}, f_n(x_{n,n-1}))| \leq M \). So \( |f'_n| \leq M \).

For any \( t \in [0, 1] \), if \( t = x_{n,i} \) for some \( i \leq n \), then \( |\Delta_n(t)| = |0| = 0 \leq 2M \). Otherwise, \( |\Delta_n(t)| = |f'_n(t) - \phi(t, f_n(t))| \leq |f'_n(t)| + |\phi(t, f_n(t))| \leq M + M = 2M \). So \( |\Delta_n| \leq 2M \).

We already showed that \( \Delta_n \in \mathcal{R} \).

We claim that for all \( t \in [0, 1] \), \( |\phi(t, f_n(t)) + \Delta_n(t)| \leq M \). If \( t = x_{n,i} \) for some \( i \leq n \), then \( |\phi(t, f_n(t)) + \Delta_n(t)| = |\phi(t, f_n(t)) + 0| \leq M \). Otherwise, \( |\phi(t, f_n(t)) + \Delta_n(t)| = |\phi(t, f_n(t)) + f'_n(t) - \phi(t, f_n(t))| \leq M \), proving the claim. So for any \( t \in [0, 1] \), we have by Theorem 6.12 (d) of Rudin that

\[
|f_n(t)| = \left| c + \int_0^x (\phi(t, f_n(t)) + \Delta_n(t)) \, dt \right| \leq |c| + \left| \int_0^x (\phi(t, f_n(t)) + \Delta_n(t)) \, dt \right| \leq |c| + M |x| \leq |c| + M.
\]
Defining $M_1 = |c| + M$, we have that $|f_n| \leq M_1$.

(b) Consider any $\varepsilon > 0$ and define $\delta = \frac{\varepsilon}{M}$. For any $x, y \in [0, 1]$ with $|x - y| < \delta$ and any $n \geq 1$ we have (assuming w.l.o.g. that $x \leq y$)

$$|f_n(x) - f_n(y)| = \left| c + \int_{0}^{x} (\phi(t, f_n(t)) + \Delta_n(t)) \, dt - \left( c + \int_{0}^{y} (\phi(t, f_n(t)) + \Delta_n(t)) \, dt \right) \right|$$

$$= \left| \int_{x}^{y} (\phi(t, f_n(t)) + \Delta_n(t)) \, dt \right| \leq M |y - x| < M\delta = M \left( \frac{\varepsilon}{M} \right) = \varepsilon.$$

Hence $\{f_n\}$ is equicontinuous.

(c) Since $[0, 1]$ is a closed and bounded subset of $\mathbb{R}$, it is compact. We have also shown that for each $n \geq 1$, $f_n$ is a continuous real-valued function on $[0, 1]$ such that $|f_n| \leq M_1 < \infty$ by part (a). In particular, each $f_n \in C([0, 1])$ and $\{f_n\}$ is pointwise (in fact, uniformly) bounded. By part (b), $\{f_n\}$ is equicontinuous. Hence by Theorem 7.25 of Rudin, there exists a subsequence $\{f_{n_k}\}$ that converges uniformly to some real-valued function $f$ on $[0, 1]$.

(d) The rectangle $R = [0, 1] \times [-M_1, M_1] \subseteq \mathbb{R}^2$ is closed, being a product of closed sets, and clearly bounded. Therefore, $R$ is compact, and so since $\phi$ is continuous on $R$ by assumption, it follows by Theorem 4.19 of Rudin that $\phi$ is uniformly continuous on $R$. Also, for every $t \in [0, 1]$, we have that $|f(t)| = \lim_{k \to \infty} |f_{n_k}(t)| \leq M_1$ since each $|f_{n_k}(t)| \leq M_1$, and so for all $k \in \mathbb{N}$, $(t, f_{n_k}(t)), (t, f(t)) \in R$.

Consider any $\varepsilon > 0$. By uniform continuity of $\phi$ on $R$, there exists $\delta > 0$ such that for any $a, b \in R$ with $\|a - b\| < \delta$, we have $|\phi(a) - \phi(b)| < \varepsilon$. Since $f_{n_k} \to f$ uniformly, there exists a $K$ such that for all $t \in [0, 1]$ and all $k \geq K$, we have $|f_{n_k}(t) - f(t)| < \delta$. Hence for any $t \in [0, 1]$ and $k \geq K$, we have that

$$\|(t, f_{n_k}(t)) - (t, f(t))\| = |f_{n_k}(t) - f(t)| < \delta,$$

and so $|\phi(t, f_{n_k}(t)) - \phi(t, f(t))| < \varepsilon$.

Hence $\phi(t, f_{n_k}(t)) \to \phi(t, f(t))$ uniformly on $[0, 1]$.

(e) Again, we have that $(t, f_n(t)) \in R$ for any $t \in [0, 1]$ and $n \geq 1$. Consider any $\varepsilon > 0$, and let $\delta > 0$ be defined as in part (d). By part (b), $\{f_n\}$ is equicontinuous, and so there exists a $\delta' > 0$ such that for all $n \geq 1$ and all $s, t \in [0, 1]$ with $|s - t| < \delta'$, we have $|f_n(s) - f_n(t)| < \frac{\delta}{\sqrt{2}}$. Since

$$\min \left\{ \frac{\delta}{\sqrt{2}}, \delta' \right\} > 0,$$

let $N$ be such that for all $n \geq N$, $\frac{1}{n} < \min \left\{ \frac{\delta}{\sqrt{2}}, \delta' \right\}$. Now consider any $t \in [0, 1]$ and any $n \geq N$. If $t = x_{n,i}$ for some $i \leq n$, then $|\Delta_n(t)| = |0| < \varepsilon$. Otherwise, choose $i < n$ such that $t \in (x_{n,i}, x_{n,i+1})$. In particular, $|t - x_{n,i}| < x_{n,i+1} - x_{n,i} = \frac{1}{n} < \frac{\delta}{\sqrt{2}}, \delta'$. So

$$\|(x_{n,i}, f_n(x_{n,i})) - (t, f_n(t))\| = \sqrt{(t - x_{n,i})^2 + (f_n(x_{n,i}) - f_n(t))^2} < \sqrt{\left( \frac{\delta}{\sqrt{2}} \right)^2 + \left( \frac{\delta}{\sqrt{2}} \right)^2} = \delta,$$

which implies that $|\Delta_n(t)| = |f_n'(t) - \phi(t, f_n(t))| = |\phi(x_{n,i}, f_n(x_{n,i})) - \phi(t, f_n(t))| < \varepsilon$.

Hence $\Delta_n \to 0$ uniformly on $[0, 1]$.

(f) Since $\Delta_n \to 0$ uniformly on $[0, 1]$ by part (e), it follows that $\Delta_{n_k} \to 0$ uniformly as well. Applying part (d) as well, it follows by the proof of Theorem 7.29 of Rudin that $\phi(t, f_{n_k}(t)) + \Delta_{n_k}(t) \to \phi(t, f(t))$ uniformly on $[0, 1]$. By Theorem 7.16 of Rudin, it follows that for all $x \in [0, 1]$,

$$f(x) = \lim_{k \to \infty} f_{n_k}(x) = \lim_{k \to \infty} \left( c + \int_{0}^{x} (\phi(t, f_{n_k}(t)) + \Delta_{n_k}(t)) \, dt \right) = c + \int_{0}^{x} \phi(t, f(t)) \, dt.$$
So we have that $f(0) = c + \int_0^0 \phi(t, f(t)) \, dt = c$. Since $t \mapsto \phi(t, f(t))$ is continuous on $[0, 1]$, it follows by Theorem 6.20 of Rudin that for any $x \in [0, 1]$,

$$f'(x) = \left( c + \int_0^x \phi(t, f(t)) \, dt \right)' = \phi(x, f(x)).$$

Hence $f$ is a solution to the initial-value problem.