Math 108A Problem Set 6 Sample Solutions

4.20. (a) Since $0 \leq d(x,z)$ for all $x \in X$ and $z \in E$, it follows that $\rho_E(x) = \inf_{z \in E} d(x,z) \geq 0$. Recall that $x \in \overline{E}$ if and only if for all $r > 0$, $B_r(x) \cap E \neq \emptyset$. So

$$
\rho_E(x) = 0 \iff \rho_E(x) \leq 0 \iff \forall r > 0 \left( \inf_{z \in E} d(x,z) < r \right) \iff \forall r > 0 \exists z \in E \left( d(x,z) < r \right)
$$

$$
\iff \forall r > 0 \left( B_r(x) \cap E \neq \emptyset \right) \iff x \in \overline{E}.
$$

(b) Since $E$ is non-empty, there exists a $z \in E$ and so $\rho_E(x) \leq d(x,z) < \infty$ for any $x \in X$. In particular, we can subtract $\rho_E(x)$ from both sides of an inequality. Consider any $x, y \in X$ and $z \in E$. Then $\rho_E(x) = \inf_{z' \in E} d(x,z') \leq d(x,z) \leq d(x,y) + d(y,z)$. Rearranging, we obtain $\rho_E(x) - d(x,y) \leq d(y,z)$. Since $z \in E$ was arbitrary, it follows that $\rho_E(x) - d(x,y) \leq \inf_{z' \in E} d(y,z') = \rho_E(y)$. This shows that $\rho_E(x) - \rho_E(y) \leq d(x,y)$. By repeating the argument with $x$ and $y$ switched, it also follows that $\rho_E(y) - \rho_E(x) \leq d(x,y)$. Hence $|\rho_E(y) - \rho_E(x)| \leq d(x,y)$ for all $x,y \in X$.

For any $\varepsilon > 0$, let $\delta = \varepsilon$. Then if $d(x,y) < \delta$, it follows that $|\rho_E(x) - \rho_E(y)| \leq d(x,y) < \delta = \varepsilon$, and so $\rho_E$ is uniformly continuous on $X$.

7.1. Let $X$ be a metric space and let $\{f_n\}$ be a uniformly convergent sequence of bounded functions on $E \subseteq X$. By Theorem 7.8 of Rudin, there exists an $N$ such that if $m,n \geq N$ and $x \in E$, then $|f_n(x) - f_m(x)| \leq 1$. For each $n \in \mathbb{N}$, since $f_n$ is bounded, let $M_n$ be real such that $|f_n(x)| < M_n$ for any $x \in E$. Then for $n \geq N$ and $x \in E$, we have

$$
|f_n(x)| = |f_n(x) - f_N(x) + f_N(x)| \leq |f_n(x) - f_N(x)| + |f_N(x)| < M_N + 1.
$$

Now define

$$
M = \max \{ M_0, \ldots, M_{N-1}, M_N + 1 \}.
$$

Then for any $n \in \mathbb{N}$ and $x \in E$, if $n < N$, then $|f_n(x)| < M_n \leq M$ and if $n \geq N$, then $|f_n(x)| < M_N + 1 \leq M$. Hence every uniformly convergent sequence of bounded functions is uniformly bounded.

**Lemma 1:** Suppose that $\{c_n\}$ is a sequence of complex numbers and there is an $N$ such that $c_n = 0$ for all $n > N$. Then $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{N} c_n$.

**Proof:** If $\{s_n\}$ is the sequence of partial sums, then for all $n \geq N$ we have $s_n = \sum_{k=0}^{n} c_k = \sum_{k=0}^{N} c_k + \sum_{k=N+1}^{n} 0 = \sum_{k=0}^{N} c_k$. Hence $\sum_{n=0}^{\infty} c_n = \lim_{n \to \infty} s_n = \sum_{k=0}^{N} c_k$. ■

**Definition:** Let $0 : \mathbb{R} \to \mathbb{R}$ be the constant zero function: $0(x) = 0$ for all $x \in \mathbb{R}$.

7.5. The function $0$ is clearly continuous. If $x \leq 0$, then $x < \frac{1}{n+1}$ for all $n \geq 1$, and so $f_n(x) = 0$, so that $f_n(x) \to 0$. If $x > 0$, then choose $N$ such that $\frac{1}{N} < x$. Then for all $n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} < x$ and so $f_n(x) = 0$, again showing that $f_n(x) \to 0$. So $f_n \to 0$.

So by uniqueness of limits, if $\{f_n\}$ converged uniformly, we must have that $f_n \to 0$ uniformly. If true, then there would exist an $N$ such that for all $n \geq N$ and $x \in \mathbb{R}$, $|f_n(x) - 0(x)| = |f_n(x)| < 1$.

But if we let $x = \frac{1}{N+1}$, then $\frac{1}{N+1} \leq x \leq \frac{1}{N}$, and so

$$
|f_N(x)| = \sin^2 \left( \frac{\pi}{x} \right) = \sin^2 \left( \left( N + \frac{1}{2} \right) \pi \right) = (\pm 1)^2 = 1 \neq 1,
$$

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a contradiction. So \( \{f_n\} \) does not converge uniformly.

For any \( x \in \mathbb{R} \) and \( n \geq 1 \), we have either \( f_n(x) = 0 \) or \( f_n(x) = \sin^2 \left( \frac{x}{n} \right) \geq 0 \). Hence \( f_n \) is non-negative. As shown in the first paragraph, there exists an \( N \) such that \( f_n(x) = 0 \) for all \( n > N \). Hence by Lemma 1, \( \sum_{n=1}^{\infty} |f_n(x)| = \sum_{n=1}^{N} f_n(x) \), and so the series \( \sum_{n=1}^{\infty} f_n(x) \) converges absolutely for all \( x \in \mathbb{R} \).

Suppose the series \( \sum_{n=1}^{\infty} f_n \) converged uniformly. Then by Theorem 7.8 of Rudin there would exist an \( N \) such that for all \( m, n \geq N \) and \( x \in \mathbb{R} \), \(|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x)| < 1\). If we take \( n = N + 1 \) and \( m = N \), then \(|f_N(x)| < 1\) for all \( x \in \mathbb{R} \). But we showed in the second paragraph that no such \( N \) existed, yielding a contradiction. So pointwise absolute convergence of \( \sum_{n=1}^{\infty} f_n \) does not imply uniform convergence.

**Lemma 2:** If \( |x_n| \leq s_n \) for all \( n \in \mathbb{N} \) and \( s_n \to 0 \), then \( x_n \to 0 \).

**Proof:** For any \( \varepsilon > 0 \), choose \( N \) such that \( s_n = |s_n| < \varepsilon \) for all \( n \geq N \). Then \( |x_n| \leq s_n < \varepsilon \) for these \( n \).

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**7.7.** We claim that \( f_n \to 0 \) uniformly. By the quotient rule, we have that

\[
f'_n(x) = \frac{1 (1 + nx^2) - 2nx(x)}{(1 + nx^2)^2} = \frac{1 - nx^2}{(1 + nx^2)^2}
\]

for all \( x \in \mathbb{R} \) and \( n \geq 1 \). If \( t > \frac{1}{\sqrt{n}} \), then

\[
f'_n(t) = \frac{1 - nt^2}{(1 + nt^2)^2} < \frac{1 - n \left(\frac{1}{\sqrt{n}}\right)^2}{(1 + n \left(\frac{1}{\sqrt{n}}\right)^2)^2} = 0,
\]

since \( (1 + nt^2)^2 \geq 1^2 > 0 \). It follows from the proof of Theorem 5.11 of Rudin that \( f_n \) is monotonically decreasing on \( \left[ \frac{1}{\sqrt{n}}, \infty \right) \), and so for all \( x \geq \frac{1}{\sqrt{n}} \), we have

\[
f_n(x) \leq f_n \left( \frac{1}{\sqrt{n}} \right) = \frac{\left(\frac{1}{\sqrt{n}}\right)}{1 + n \left(\frac{1}{\sqrt{n}}\right)^2} = \frac{1}{2\sqrt{n}}.
\]

Similarly, if \( 0 < t < \frac{1}{\sqrt{n}} \), then \( f'_n(t) > 0 \), and so \( f_n \) is monotonically increasing on \( \left[ 0, \frac{1}{\sqrt{n}} \right) \). So if \( 0 \leq x \leq \frac{1}{\sqrt{n}} \), then \( f_n(x) \leq \frac{1}{2\sqrt{n}} \). If \( x \geq 0 \), then \( |f_n(x)| = \left| \frac{x}{1 + nx^2} \right| = \frac{x}{1 + nx^2} = f_n(x) \) since \( 1 + nx^2 \geq 1 > 0 \). Hence for \( x \geq 0 \), we have \(|f_n(x)| = f_n(x) \leq \frac{1}{2\sqrt{n}} \). If \( x < 0 \), then \(|f_n(x)| = \left| \frac{x}{1 + nx^2} \right| = \frac{-x}{1 + nx^2} = f_n(-x) \leq \frac{1}{2\sqrt{n}} \) as well. By Theorem 3.20 (a) of Rudin, \( \frac{1}{\sqrt{n}} \to 0 \), and so for any \( \varepsilon > 0 \), there exists an \( N \) such that \( \frac{1}{2\sqrt{n}} < \varepsilon \) for all \( N \geq N \). So for any \( n \geq N \) and \( x \in \mathbb{R} \), we have that

\[
|f_n(x) - 0(x)| \leq \frac{1}{2\sqrt{n}} < \varepsilon,
\]

and so \( f_n \to 0 \) uniformly.

We have \( f'(x) = 0 \) for all \( x \in \mathbb{R} \). If \( x \neq 0 \), then

\[
|f'_n(x)| = \left| \frac{1 - nx^2}{(1 + nx^2)^2} \right| \leq \frac{|1| + |nx^2|}{(1 + nx^2)^2} = \frac{1}{1 + nx^2} < \frac{1}{x^2n}.
\]
Since \( \lim_{n \to \infty} \frac{1}{x^n} = 0 \), it follows by Lemma 2 that \( f_n'(x) \to 0 \). On the other hand, we have that 
\[
f_n'(0) = \frac{1-n(0)^2}{(1+n(0))^2} = 1 \text{ for all } n \in \mathbb{N},
\]
and so \( f_n'(0) \to 0 \). Hence the equation \( 0'(x) = \lim_{n \to \infty} f_n'(x) \) is correct if \( x \neq 0 \) but false when \( x = 0 \).

**Lemma 3:** Suppose that \( a_n \geq 0 \) for all \( n \in \mathbb{N} \) and \( \sum_{n=0}^{N} a_n \leq a \) for all \( N \in \mathbb{N} \). Then \( \sum_{n=0}^{\infty} a_n \) converges and \( \sum_{n=0}^{N} a_n \leq \sum_{n=0}^{\infty} a_n \) for all \( N \in \mathbb{N} \).

**Proof:** Since each \( a_n \geq 0 \), the sequence \( \{s_n\} \) of partial sums is monotonically increasing, and \( \{s_n\} \) is bounded above by \( a \) by assumption. Hence the proof of Theorem 3.14 of Rudin shows that \( \sum_{n=0}^{\infty} a_n = \sup_{N \in \mathbb{N}} \sum_{n=0}^{N} a_n \) converges.

**Lemma 4:** Suppose that \( x \in [0,1] \). Then there exists a sequence \( \{a_n\} \) in \( \{0,1\} \) such that \( x = \sum_{n=1}^{\infty} 2^{-n}a_n \).

**Proof:** Define \( a_1 = 1 \) if \( x \geq \frac{1}{2} \) and \( a_1 = 0 \) otherwise. Having defined \( a_1, \ldots, a_n \), let \( a_{n+1} = \max \{ j < 2 : \sum_{i=1}^{n} 2^{-i}a_i + 2^{-(n+1)}j \leq x \} \). Then by definition, we have that \( \sum_{i=1}^{n} 2^{-i}a_i \leq x \) for all \( n \geq 1 \), and so \( x \geq \sum_{n=1}^{\infty} 2^{-n}a_n \) (which converges by Lemma 3). We claim that \( x - \sum_{i=1}^{n} 2^{-i}a_i \leq 2^{-n} \) for all \( n \geq 1 \). This is immediate for \( n = 1 \). If true for \( n \), then \( x - \sum_{i=1}^{n} 2^{-i}a_i \leq 2^{-n} - 2^{-(n+1)}a_{n+1} \leq 2^{-(n+1)} \) if \( a_{n+1} = 1 \). If, toward a contradiction, \( a_{n+1} = 0 \) and \( x - \sum_{i=1}^{n} 2^{-i}a_i > 2^{-(n+1)} \), then \( \sum_{i=1}^{n} 2^{-i}a_i + 2^{-(n+1)} < x \), contradicting that \( a_{n+1} = 0 \). So for any \( \varepsilon > 0 \), we can choose \( N \) with \( 2^{-N} < \varepsilon \) and see (by Lemma 3) that \( x - \sum_{n=1}^{\infty} 2^{-n}a_n \leq x - \sum_{i=1}^{N} 2^{-i}a_i \leq 2^{-N} < \varepsilon \). Hence \( x = \sum_{n=1}^{\infty} 2^{-n}a_n \).

**Lemma 5:** Suppose \( f \) is a function on \( \mathbb{R} \) such that there exists \( b \in \mathbb{R} \) with \( f(t + b) = f(t) \) for all \( t \in \mathbb{R} \). Then for any integer \( k \in \mathbb{Z} \), \( f(t + kb) = f(t) \).

**Proof:** We prove this for \( k \geq 0 \) by induction. It is immediately true for \( k = 0 \). If true for \( k \), then \( f(t + (k + 1)b) = f(t + kb + b) = f(t + b) = f(t) \) by assumption on \( f \). Similarly, we can use induction to prove it for \( k < 0 \).

7.14. Since \( f \) is a continuous function on \( \mathbb{R} \), it follows from Theorems 4.7 and 4.9 of Rudin that for all \( n \geq 1 \), \( t \mapsto 2^{-n}f(3^{2n-1}t) \) is continuous as well, and hence so are the functions \( g_N \) defined by 
\[
g_N(t) = \sum_{n=1}^{N} 2^{-n}f(3^{2n-1}t)
\]
for all \( t \in \mathbb{R} \). By definition, we have \( x = \lim_{N \to \infty} g_N \). Since \( 0 \leq f \leq 1 \), we have that \( |2^{-(n+1)}f(3^{2n}t)| = 2^{-n}f(3^{2n}t) \leq 2^{-n} \) for all \( t \in \mathbb{R} \). Since \( \sum_{n=1}^{\infty} 2^{-n} = 1 \) converges, it follows from Theorem 7.10 of Rudin that \( x(t) = \sum_{n=1}^{\infty} 2^{-n}f(3^{2n-1}t) \) converges uniformly on \( \mathbb{R} \).

Since \( g_N \to x \) uniformly on \( \mathbb{R} \) and each \( g_N \) is continuous on \( \mathbb{R} \), it follows from Theorem 7.12 of Rudin that \( x \) is continuous on \( \mathbb{R} \). By exactly the same argument, \( y \) is also continuous on \( \mathbb{R} \). Hence by Theorem 4.10 (a) of Rudin, \( \Phi(t) = (x(t), y(t)) \) is continuous on \( \mathbb{R} \).

Since \( 0 \leq f \leq 1 \), we have that 
\[
0 = \sum_{n=1}^{\infty} 0 \leq \sum_{n=1}^{\infty} 2^{-n}f(3^{2n-1}t) \leq \sum_{n=1}^{\infty} 2^{-n} = 1
\]
for all \( t \in \mathbb{R} \). Similarly, \( y(t) \in I \) for all \( t \in \mathbb{R} \). Hence \( \Phi(t) = (x(t), y(t)) \in I^2 \) for all \( t \in \mathbb{R} \). Now consider any \( (x_0, y_0) \in I^2 \). By Lemma 4, let \( \{b_n\} \) and \( \{c_n\} \) be sequences in \( \{0,1\} \) such that \( x_0 = \sum_{n=1}^{\infty} 2^{-n}b_n \) and \( y_0 = \sum_{n=1}^{\infty} 2^{-n}c_n \), and let \( a_{2n-1} = b_n \) and \( a_{2n} = c_n \) for all \( n \geq 1 \). Let \( t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i) \). Then (as argued in the sample solutions to homework 4), \( t_0 \) converges and \( t_0 \in P \), the Cantor set, since each \( 2a_i \in \{0,2\} \). Then for any \( k \geq 1 \), we have 
\[
3^k t_0 = 3^k \sum_{i=1}^{\infty} 3^{-i-1}(2a_i) = \sum_{i=1}^{\infty} 3^{k-i-1}(2a_i) = 2 \sum_{i=1}^{k-1} 3^{k-i-1}a_i + \frac{2a_k}{3} + \sum_{i=k+1}^{\infty} 3^{k-i-1}(2a_i).
\]
Because it is a sum of integers, \( \sum_{i=1}^{k-1} 3^{k-i-1}a_i \) is an integer. We also have \( 0 = \sum_{i=k+1}^{\infty} 3^{k-i-1} (2a_i) \leq \sum_{i=k+1}^{\infty} 2 \cdot 3^{k-i-1} = \frac{1}{3} \). So if \( a_k = 1 \), then by Lemma 5,

\[
\begin{align*}
\Phi(t_0) &= (x(t_0), y(t_0)) = \left( \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t_0), \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t_0) \right) \\
&= \left( \sum_{n=1}^{\infty} 2^{-n}a_{2n-1}, \sum_{n=1}^{\infty} 2^{-n}a_{2n} \right) \\
&= \left( \sum_{n=1}^{\infty} 2^{-n}b_n, \sum_{n=1}^{\infty} 2^{-n}c_n \right) = (x_0, y_0).
\end{align*}
\]

This shows that \( I^2 \subseteq \Phi(P) \), and we already showed \( \Phi(P) \subseteq \Phi(\mathbb{R}) \subseteq I^2 \). Hence \( \Phi \) maps \( P \) onto the unit square \( I^2 \). Since \( I^2 \subseteq \Phi(P) \subseteq \Phi(I) \), we also have \( \Phi(I) = I^2 \) and \( \Phi \) maps \( I \) onto \( I^2 \).