WEEK 6: REAL FUNCTIONS

Example 1. Find all continuous functions \( f : \mathbb{R} \to \mathbb{R} \) satisfying

\[
f(x + y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.
\]

Example 2. Find all continuous functions \( f : \mathbb{R} \to \mathbb{R} \) that satisfy the relation

\[
3f(2x + 1) = f(x) + 5x \quad \text{for all } x \in \mathbb{R}.
\]

Example 3. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous and strictly decreasing function. Prove that the equation \( f(f(x)) = x \) has a unique solution.

Example 4. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( |f(x) - f(y)| \geq |x - y| \) for all \( x, y \in \mathbb{R} \). Prove that \( f \) is surjective; in other words, the range of \( f \) is all of \( \mathbb{R} \).

Example 5. Find all bounded functions \( f : \mathbb{R} \to \mathbb{R} \) such that

\[
f(f(f(x))) = 3f(f(x)) - 3f(x) + x
\]

for all real numbers \( x \).