WEEK 3: NUMBER THEORY

Example 1. Let \( a_1, a_2, \ldots \) be a sequence such that \( a_1 = 43, a_2 = 142, \) and \( a_{n+1} = 3a_n + a_{n-1} \) for all \( n \geq 2. \)
Prove that

(a) \( a_n \) and \( a_{n+1} \) are relatively prime for all \( n \geq 1; \)
(b) for every positive integer \( m, \) there exist infinitely many \( n \)'s such that \( a_n - 1 \) and \( a_{n+1} - 1 \) are both divisible by \( m. \)

Example 2. Suppose that \( \{a_n\}_{n \geq 1} \) is a sequence of positive integers satisfying \( \gcd(a_i, a_j) = \gcd(i, j) \) for \( i \neq j. \) Show that \( a_n = n \) for each \( n. \)

Example 3. Let \( f \) be a bijection of the set of positive integers. Prove that there exist positive integers \( a < a + d < a + 2d \) such that \( f(a) < f(a + d) < f(a + 2d). \)

Example 4. For each positive integer \( n, \) let \( f(n) \) be the greatest common divisor of \( 100+n^2 \) and \( 100+(n+1)^2. \)
Find the maximum value of \( f(n) \) as \( n \) ranges through the positive integers.

Example 5. Let \( \{x_n\}_{n \geq 1} \) be a sequence of nonnegative integers such that

(i) \( x_1 = x_4 = x_5 = 1, x_2 = x_3 = x_6 = 0; \)
(ii) \( x_{n+6} \) equals the last digit of \( x_n + x_{n+1} + \cdots + x_{n+5} \) in decimal expansion for all \( n. \)
Does there exists a positive integer \( n \) such that \( x_n = x_{n+2} = x_{n+4} = 0 \) and \( x_{n+1} = x_{n+3} = x_{n+5} = 1? \)