PROBLEM SET 7

Rules: Submit your solution to at least one of the following problems by Tuesday, November 15 in class, or in my mailbox at Sloan. It is perfectly acceptable to submit partial solutions explaining what progress you have made, or anything else that shows that you gave some thought to the problems.

There are two types of problems. Students with little experience on problem solving are recommended to first try type A problems. Most of type B problems appeared in the Putnam competition in past years.

A1. Evaluate the following integrals:

(1) \[ \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}} \, dx \]

(2) \[ \int_0^1 \frac{dx}{x + \sqrt{1-x^2}} \]

A2. Prove the Caushy-Swartz inequality for continuous functions \( f, g \) on \([0,1] \):

\[ \left( \int_0^1 f(t)^2 \, dt \right) \left( \int_0^1 g(t)^2 \, dt \right) \geq \left( \int_0^1 f(t)g(t) \, dt \right)^2. \]

A3. Find the minimum value of \( \int_0^1 f(x)(x - f(x)) \, dx \)

as \( f \) ranges over all continuous real-valued functions on \([0,1] \).

A4. Find all positive continuous function \( f \) on \([0,1] \) such that

\[ \int_0^1 f(x) \, dx = 1, \quad \int_0^1 x f(x) = \alpha, \quad \int_0^1 x^2 f(x) = \alpha^2 \]

for some positive number \( \alpha \).
B 1. (2014 B2) Suppose that $f$ is a function on the interval $[1, 3]$ such that $-1 \leq f(x) \leq 1$ for all $x$ and $\int_{1}^{3} f(x) \, dx = 0$. How large can $\int_{1}^{3} \frac{f(x)}{x} \, dx$ be?

B 2. (2009 B2) A game involves jumping to the right on the real number line. If $a$ and $b$ are real numbers and $b > a$, the cost of jumping from $a$ to $b$ is $b^3 - ab^2$. For what real numbers $c$ can one travel from 0 to 1 in a finite number of jumps with total cost exactly $c$?

B 3. (2006 B5) For each continuous function $f : [0, 1] \to \mathbb{R}$, let $I(f) = \int_{0}^{1} x^2 f(x) \, dx$ and $J(x) = \int_{0}^{1} x (f(x))^2 \, dx$. Find the maximum value of $I(f) - J(f)$ over all such functions $f$.

B 4. (2005 A5) Evaluate

$$\int_{0}^{1} \frac{\ln(x + 1)}{x^2 + 1} \, dx.$$