PROBLEM SET 5

Rules: Submit your solution to at least one of the following problems by Tuesday, November 1 in class, or in my mailbox at Sloan. It is perfectly acceptable to submit partial solutions explaining what progress you have made, or anything else that shows that you gave some thought to the problems.

There are two types of problems. Students with little experience on problem solving are recommended to first try type A problems. Most of type B problems appeared in the Putnam competition in past years.

A1. Find all polynomials $P(x)$ with real coefficients such that $(x + 1)P(x) = (x - 2)P(x + 1)$ for all real numbers $x$.

A2. Determine all polynomials $P(x)$ with real coefficients for which there exists a positive integer $n$ such that for all real numbers $x$,
$$P \left( x + \frac{1}{n} \right) + P \left( x - \frac{1}{n} \right) = 2P(x).$$

A3. Let $P(x)$ be a polynomial with integer coefficients. Assume that there exist three different integers $a, b, c$ such that $P(a) = P(b) = P(c) = -1$. Prove that $P(x)$ has no integer roots.

A4. Let $P(x)$ be a polynomial with complex coefficients. Prove that $P(x)$ is an even function if and only if there exists a polynomial $Q(x)$ with complex coefficients satisfying
$$P(x) = Q(x)Q(-x).$$
**B1.** (2005 B1) Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers $a$. (Note: $\lfloor v \rfloor$ is the greatest integer less than or equal to $v$.)

**B2.** (1999 B2) Let $P(x)$ be a polynomial of degree $n$ such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have $n$ distinct roots.

**B3.** (2003 B4) Let

$$f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$$

where $a, b, c, d, e$ are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number and $r_1 + r_2 \neq r_3 + r_4$, then $r_1r_2$ is a rational number.

**B4.** (2010 B4) Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x + 1) - p(x + 1)q(x) = 1.$$