PROBLEM SET 3

Rules: Submit your solution to at least one of the following problems by Tuesday, October 11 in class, or in my mailbox at Sloan. It is perfectly acceptable to submit partial solutions explaining what progress you have made, or anything else that shows that you gave some thought to the problems.

There are two types of problems. Students with little experience on problem solving are recommended to first try type A problems. Most of type B problems appeared in the Putnam competition in past years.

A1. Let $n$ be a positive integer. Suppose that $2^n$ and $5^n$ begin with the same digit. Then there is only one possible value for this common initial digit.

A2. A positive number $n$ is said to be good if it can be written as a sum of two or more consecutive positive integers. For example, $3 = 1 + 2$ and $15 = 4 + 5 + 6$ are good numbers. Determine the number of good numbers that are less than or equal to 2015.

A3. For a positive integer $n$, define $f(n) = \max_{m \in \{1, 2, \ldots, n\}} d(m)$ where $d(m)$ denotes the number of positive divisors of $m$. Prove that there are infinitely many integers $k$ such that there does not exist a positive integer $n$ with $f(n) = k$.

A4. (1) Let $a$ and $b$ be positive integers, and define a sequence $\{x_n\}_{n \geq 0}$ by $x_0 = 1$ and $x_{n+1} = ax_n + b$ for all $n \geq 0$. Prove that infinitely many terms of the sequences are composite.

(2) Let $\{y_n\}_{n \geq 0}$ be a sequence of positive integers satisfying the recurrence relation $y_{n+1} = 5y_n - 6y_{n-1}$. Prove that infinitely many terms of the sequence are composite.
B1. (2009 B1) Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

\[
\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}
\]

B2. (2005 A1) Show that every positive integer is a sum of one or more numbers of the form $2^r3^s$, where $r$ and $s$ are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

B3. (2013 A2) Let $S$ be the set of all positive integers that are not perfect squares. For $n$ in $S$, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of $a_r$ over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3, 2 \cdot 4, 2 \cdot 5, 2 \cdot 3 \cdot 4, 2 \cdot 3 \cdot 5, 2 \cdot 4 \cdot 5,$ and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function $f$ from $S$ to the integers is one-to-one.

B4. (2006 A3) Let $1, 2, 3, \cdots, 2005, 2006, 2007, 2009, 2012, 2016, \cdots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \cdots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.