PROBLEM SET 1

**Rules:** Submit your solution to at least one of the following problems by **Tuesday, October 4** in class, or in my mailbox at Sloan. It is perfectly acceptable to submit partial solutions explaining what progress you have made, or anything else that shows that you gave some thought to the problems.

There are two types of problems. Students with little experience on problem solving are recommended to first try type A problems. Most of type B problems appeared in the Putnam competition in past years.

**A1.** Do Example 1 and 2 with the “misère” version of Nim games. That is, the player who makes the last move loses.

**A2.** Two players take turns writing positive integers on the board. The first number must be 1; the number written on each turn must be between $2^N$ and $10^N$, inclusive, where $N$ is the number written on the previous turn. The goal is to be the first to write a number greater than or equal to $10^6$. Who wins?

**A3.** There are $n > 1$ toothpicks in a row. The first player can take up to $n - 1$ toothpicks. The second player can take at most the number of toothpicks that the first has just taken. The first can then take at most the number of toothpicks that the second has just taken. This continues until all the toothpicks are gone. The player who takes the last toothpick wins. For which $n$ does the first player have a winning strategy?

**A4.** Nine cards are on the table, numbered one through nine. The two players alternate picking up cards. The first player to have three cards summing to fifteen wins. If all cards are picked up without either player winning, the game is declared a draw. Assuming that both players play perfectly, determine the outcome of the game.
B1. (2002 A4) In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty $3 \times 3$ matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the $3 \times 3$ matrix is completed with five 1’s and four 0’s. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

B2. (1995 B5) A game starts with four heaps of beans, containing 3, 4, 5, and 6 beans. The two players move alternately. A move consists of taking either
(a) one bean from a heap, provided at least two beans are left behind in that heap, or
(b) a complete heap of two or three beans.
The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

B3. (2013 B6) Let $n \geq 1$ be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of $n$ spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either
- places a stone in an empty space, or
- removes a stone from a nonempty space $s$, places a stone in the nearest empty space to the left of $s$ (if such a space exists), and places a stone in the nearest empty space to the right of $s$ (if such a space exists).
Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?