Week 7 Summary

Last week we saw how painful it can be to work directly with the definition of integrability to show that a function is integrable and/or to find the integral of a function. We will take a look at a few tools including the First and the Second Fundamental Theorems of Calculus which allow us to work with integrals without having to use the ε-definition.

Then, we will learn how we can approximate integrals using Taylor series and numerical methods. As with Newton’s method, numerical approximation is very useful in the real world when it is hard (or even impossible) to find the true value.

Topics

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1 Tools

We just saw how difficult it can be to work directly with the definition of integrability. Even with simple functions such as \( f(x) = x^p \), showing that it is integrable and computing the integral over an interval was extremely nontrivial.

Fortunately, we have tools that allow us to get by without dealing with the definition directly.

**Proposition 1.1** (Arithmetic of integrals). Let \( f \) and \( g \) be functions that are integrable on an interval \([a,b]\). Then,

1. For all \( \alpha, \beta \in \mathbb{R} \), we have
   \[
   \int_a^b \alpha f(x) + \beta g(x) \, dx = \alpha \int_a^b f(x) \, dx + \beta \int_a^b g(x) \, dx
   \]

2. If \( f(x) \leq g(x) \) for all \( x \in [a,b] \), then
   \[
   \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx
   \]

3. If \( f \) is integrable on \([a,c]\) and \( a \leq b \leq c \), then
   \[
   \int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx
   \]

As was the case with limits of sequences and functions, being able to simply add integrals this way is very useful and saves us a lot of work.

Now, we look at the **Fundamental Theorem of Calculus**
Theorem 1.1 (Second Fundamental Theorem of Calculus). Let $F(x)$ be continuous on the interval $[a, b]$ with $F'(x) = f(x)$ (then we say that $F$ is the antiderivative of $f$). If $f$ is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Example 1.1. Use the Second Fundamental Theorem of Calculus to compute the integral of $f(x) = x^p$ over $[0, b]$.

Solution. We already showed that $f(x)$ is integrable on $[0, b]$ for any $b > 0$. Now, note that if we let $F(x) = \frac{x^{p+1}}{p+1}$, then we have $F'(x) = f(x)$. Hence, by Theorem 1.1, we conclude that

$$\int_0^b f(x)\,dx = F(b) - F(0) = \frac{b^{p+1}}{p+1}$$

Theorem 1.2 (First Fundamental Theorem of Calculus). Let $f$ be continuous on $[a, b]$ and define

$$F(x) = \int_a^x f(y)\,dy$$

Then, $F'(x) = f(x)$.

Corollary 1.1 (Integration by Parts). Let $f$ and $g$ be continuous functions with antiderivatives $F$ and $G$, respectively. Then,

$$\int f(x)G(x)\,dx = F(x)G(x) - \int F(x)g(x)\,dx$$

Proof. You might know integration by parts from high school. It’s a very useful trick when the integrand is a product of two simpler functions. Proof only uses the First Fundamental Theorem of Calculus and the product rule.

By the product rule, we know that

$$(F(x)G(x))' = F'(x)G(x) + F(x)G'(x) = f(x)G(x) + F(x)g(x)$$

Now, by the First Fundamental Theorem of Calculus, we have

$$F(x)G(x) = \int (F(x)G(x))'\,dx = \int f(x)G(x) + F(x)g(x)\,dx = \int f(x)G(x)\,dx + \int F(x)g(x)\,dx$$

By rearranging terms, we obtain the statement of Corollary 1.1:

$$\int f(x)G(x)\,dx = F(x)G(x) - \int F(x)g(x)\,dx$$

For our final tool, we need to define uniform continuity.

Definition (Uniform Continuity)

Let $f$ be a function on an interval $I$. We say that $f$ is uniformly continuous on $I$ if for any $\varepsilon > 0$, we can find $\delta > 0$ such that for any $x, y \in I$ satisfying $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$. 

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The definition of uniform continuity looks very much like the definition of continuity with one crucial difference: Continuity of $f$ was defined at each fixed point $a$, whereas uniform continuity of $f$ is defined over the entire interval $I$.

But how are the two notions related? It turns out that uniform continuity is a strictly stronger condition than regular continuity.

**Proposition 1.2.** Let $f$ be uniformly continuous on $I$. Then, $f$ is continuous on $I$.

What about the converse? It turns out that the converse to the Proposition 1.2 is false.

**Example 1.2.** Consider $f(x) = \frac{1}{x}$ on $I = (0, 1)$. Then, $f$ is continuous but not uniformly continuous on $I$.

The proof will be left as an exercise. It’s not too hard but definitely nontrivial to show that $f(x)$ is not uniformly continuous on $(0, 1)$.

But do not despair. The converse, while not true, is “very close to being true.”

**Theorem 1.3.** Let $f$ be a continuous function on an interval $I$. Moreover, suppose $I$ is a closed and bounded interval. Then, $f$ is uniformly continuous on $I$.

This theorem is great. Now, we know that if we are given a closed an bounded interval (for example $[0, 1], [a, b]$, etc) continuity is equivalent to uniform continuity. But why do we care about uniform continuity?

**Theorem 1.4.** Let $f$ be a uniformly continuous function on $I$. Then, $f$ is integrable on $I$.

Theorem 1.4 will save us so much potential trouble by letting us bypass the $\varepsilon$-definition of integrability. For example recall that in the proof of Example 1.1 using the Second Fundamental Theorem of Calculus, we still needed to show that $x^p$ was integrable on the interval $[0, b]$. Proof that $x^p$ is integrable on $[0, b]$ was quite painful: we had to write out $U_{[0, b]}$, $L_{[0, b]}$, compute their difference and show that the difference can be made small.

However, now we can use Theorem 1.4 to bypass all that.

**Proof.** (Showing $x^p$ is integrable on $[0, b]$)

Note that $x^p$ is continuous on any interval of the form $[0, b]$. However, since $[0, b]$ is a closed and bounded interval, $x^p$ is uniformly continuous on $[0, b]$. Hence, by Theorem 1.4, we conclude that $x^p$ is integrable on $[0, b]$. \(\square\)

## 2 Approximating Integrals using Taylor Polynomials

### 2.1 Definitions

When we first defined the derivative, recall that it was supposed to be the “instantaneous rate of change” of a function $f(x)$ at a given point $c$. In other words, $f'$ gives us a linear approximation of $f(x)$ near $c$: for small values of $\varepsilon \in \mathbb{R}$, we have

$$f(c + \varepsilon) \approx f(c) + \varepsilon f'(c)$$

But if $f(x)$ has higher order derivatives, why stop with a linear approximation? Taylor series take this idea of linear approximation and extends it to higher order derivatives, giving us a better approximation of $f(x)$ near $c$.

**Definition** (Taylor Polynomial and Taylor Series)

Let $f(x)$ be a $C^n$ function i.e. $f$ is $n$-times continuously differentiable. Then, the $n$-th order Taylor polynomial of $f(x)$ about $c$ is:

$$T_n(f)(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k$$
The $n$-th order remainder of $f(x)$ is:

$$R_n(f)(x) = f(x) - T_n(f)(x)$$

If $f(x)$ is $C^\infty$, then the Taylor series of $f(x)$ about $c$ is:

$$T_\infty(f)(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k$$

Note that the first order Taylor polynomial of $f(x)$ is precisely the linear approximation we wrote down in the beginning.

Now that we defined Taylor polynomials as higher order extensions of the linear approximation, we have to justify our claim that Taylor polynomials are indeed approximations. So what does it mean for a Taylor polynomial $T_n(f)(x)$ to be a good approximation of $f(x)$? It means that $T_n(f)(x)$ should be close to the true value of $f(x)$. In other words, we want $f(x) - T_n(f)(x)$ to be close to 0. But we defined this difference to be something...

**Theorem 2.1.** Let $R_n(f)(x)$ be the $n$-th order remainder of $f(x)$. Then, $R_n(f)(x)$ is $o((x-c)^n)$.

Theorem 2.1 is saying precisely that $T_n(f)(x)$ is very close to the real value of $f(x)$ when $x$ is near $c$. Hence, we have our justification for calling Taylor polynomials “higher order approximations” of $f(x)$. Now, we look at another very useful theorem, which will actually let us compute $R_n(f)(x)$.

**Theorem 2.2.** Suppose $f(x)$ is $(n+1)$-times continuously differentiable. Then,

$$R_n(f)(x) = \int_c^x \frac{f^{(n+1)}(c)}{n!} (x-y)^n dy$$

With Theorem 2.2, we will be able to know exactly by how much $T_n(f)(x)$ is off the true value of $f(x)$.

### 2.2 Examples

**Example 2.1.** Compute the Taylor series for $f(x) = e^x$ about 0.

**Solution.** Recall that the Taylor series of $f(x)$ is simply

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

However, for all $k$, $f^{(k)}(x) = e^x$. Hence, for all $k$, $f^{(k)}(0) = e^0 = 1$. Therefore,

$$e^x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

as we already know. 

Here are the Taylor series about 0 for some of the functions that we have come across several times. Try to do a couple of them as an exercise!

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$
Let’s look closely at the Taylor series for \( \sin x \) and \( \cos x \). It looks like we’ve split up the Taylor series of \( e^x = 1 + x + \frac{x^2}{2!} + \cdots \) into two and alternated signs. So can we find any relation between these three Taylor series?

The answer is yes and in fact, we will see something amazing come out of the inspection. Let \( i \) be the imaginary number. If you have never seen \( i \) before, it’s just some “number” (not real) with the property that \( i^2 = -1 \). As baffling as it might be to raise a number to a complex power, let’s take a leap of faith and raise \( e \) to \( ix \) power and see what we get. Plugging in \( ix \) in the Taylor series for \( e^x \) we get:

\[
e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots
\]

\[
e^{ix} = 2 + ix - \frac{x^2}{2!} + \frac{ix^4}{4!} + \frac{x^5}{5!} - \cdots
\]

\[
e^{ix} = \left( 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right) + \left( ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} + \cdots \right)
\]

\[
e^{ix} = \left( 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right) + i \left( ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} + \cdots \right)
\]

\[
e^{ix} = \left( \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \right) + i \left( \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \right)
\]

\[
e^{ix} = \cos x + i \sin x
\]

So there we have it!

\[e^{ix} = \cos x + i \sin x\]

If you are still not convinced that this formula is the most beautiful thing you’ve seen in math, try plugging in \( x = \pi \). If we let \( x = \pi \), what do we get?

\[e^{i\pi} = \cos \pi + i \sin \pi = -1 \implies e^{i\pi} + 1 = 0\]

This equation is called the Euler’s formula and dubbed “the most beautiful equation in mathematics”. 0, 1, \( \pi \), e, and \( i \) are arguably the five most important numbers in all of math and to see them appear in one equation is indeed quite amazing!

### 2.3 Approximating Integrals

Now, we will see how Taylor polynomials can help us approximate integrals. For example, consider the Gaussian integral \( \int e^{-x^2} \, dx \) called the Gaussian for short. The Gaussian is a very important integral, one of the properties being that it is the curve that represents the normal distribution a.k.a. the bell curve.

We would like to evaluate the Gaussian but there is one problem: there is no elementary antiderivative of \( e^{-x^2} \). This means that we cannot rely on the Fundamental Theorem of Calculus to evaluate the integral. But using Taylor series, we can approximate the value of this integral.

**Example 2.2.** Approximate \( \int_0^{\frac{1}{2}} e^{-x^2} \, dx \) to within \( 10^{-6} \) of its actual value.

**Solution.** To simplify notation, we will write \( T_n(x) \) and \( R_n(x) \) for \( T_n(e^{-x^2})(x) \) and \( R_n(e^{-x^2})(x) \), respectively.

For any \( n \), we have \( e^{-x^2} = T_n(x) + R_n(x) \). By integrating both sides, we obtain

\[
\int_0^{\frac{1}{2}} e^{-x^2} \, dx = \int_0^{\frac{1}{2}} T_n(x) \, dx + \int_0^{\frac{1}{2}} R_n(x) \, dx
\]
Now, $T_n(x)$ is just a polynomial. Therefore, $\int_0^{\frac{1}{3}} T_n(x)dx$ is an integral that we can explicitly compute. On the other hand, we know that $R_n(x)$ goes to 0 as $n$ increases. So the idea is to make $|\int R_n(x)dx|$ small by increasing $n$: In our case, we want to find $n$ such that $|\int_0^{1/3} R_n(x)dx| < 10^{-6}$.

By Theorem 2.2, we have

$$R_n(e^{-x})(x) = \int_0^x (-1)^{n+1} \frac{e^{-y}}{n!} (x-y)^n dy$$

However, note that

$$R_n(x) = e^{-x^2} - T_n(x) = e^{-x^2} - \left(\sum_{k=0}^{n} \frac{(-x^2)^k}{k!}\right) = R_n(e^{-x})(x^2)$$

Hence, we see that

$$R_n(x) = \int_0^{x^2} (-1)^{n+1} \frac{e^{-y}}{n!} (x^2-y)^n dy$$

Unfortunately, this is not something we can easily integrate. However, we are not interested in the actual value of the integral. We are only interested in making this integral close to 0.

How do we bound $R_n(x)$? First, note that for any $y \in [0, x^2]$, $e^{-y} \leq e^0 = 1$; and $(x^2 - y)^n \leq (x^2 - 0)^n = x^{2n}$. Note also that for all $y \in [0, x^2]$, we have $\frac{2^ny}{n!} (x^2 - y)^n \geq 0$. This gives us

$$|R_n(x)| = \left| \int_0^{x^2} (-1)^{n+1} \frac{e^{-y}}{n!} (x^2-y)^n dy \right|$$

$$= \int_0^{x^2} \frac{e^{-y}}{n!} (x^2-y)^n dy$$

$$\leq \int_0^{x^2} \frac{1}{n!} x^{2n} dy$$

$$= \left[ \frac{x^{2n}}{n!} y \right]_0^{x^2}$$

$$= \frac{x^{2n+2}}{n!} (1)$$

In our case, we want

$$\left| \int_0^{\frac{1}{3}} R_n(y)dy \right| < 10^{-6}$$

Therefore, we need to find a value of $n$ for which

$$\frac{1}{n!} \left( \frac{1}{3} \right)^{2n+2} < 10^{-6}$$

Checking small values of $n = 1, 2...$ we see that when $n = 3$, the inequality is satisfied.

This means that $\int_0^{\frac{1}{3}} T_3(x)dx$ is within $10^{-6}$ of the true value of $\int_0^{\frac{1}{3}} e^{-x^2} dx$. First, let’s compute $T_3(x)$ which is easy since we already know that Taylor series of $e^x$:

$$T_3(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!}$$
We can now integrate $T_3(x)$ over $[0, \frac{1}{3}]$ to approximate the Gaussian:

$$\int_0^{\frac{1}{3}} T_3(x) \, dx = \int_0^{\frac{1}{3}} 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} \, dx = \left[ x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} \right]_0^{\frac{1}{3}} = \frac{147604}{459270}$$

Therefore, we conclude that

$$\int_0^{\frac{1}{3}} e^{-x^2} \, dx = \frac{147604}{459270} \pm 10^{-6}$$

This example shows that Taylor polynomials can be used effectively to approximate integrals. However, we should note that approximating with Taylor polynomials works well near the point about which we are writing the Taylor polynomial. For example, if we were to approximate $\int_0^{2} e^{-x^2} \, dx$ to within $10^{-1}$ of its true value using Taylor polynomials, we would need to compute $\int_0^{2} T_{11}(x) \, dx$.

In our example, the third order Taylor polynomial was good enough to approximate the integral to within $10^{-6}$. However, as we get farther away from 0 (for us from $\frac{1}{3}$ to 2), we need the eleventh order Taylor polynomial just to get a value that is within $10^{-1}$ of the true value.