**Week 1 Notes Summary**

This week, we will learn about mathematical proofs. We start by defining what we mean by a mathematical proof and look at a few important things to avoid/keep in mind when writing mathematical proofs. Then, we will learn various techniques that can be used to prove mathematical statements. Namely, we will look at direct proofs, proofs by contradiction, and proofs by induction.

**Topics**

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**1 Introduction**

- Course: Ma 8
- Instructor: Seunghee Ye
- Email: syye@caltech.edu
- Course webpage: since you’re reading this, I assume you’ve found the course webpage

Ma 8 is a course that is designed to help you transition from high school math to Caltech math. All courses at Caltech are there to help you learn but that is especially the case for this course. The sole purpose of Ma 8 is to help you with the sometimes difficult transition to the world of “rigorous” mathematics. As such, I welcome all questions. There are no dumb questions and if something is not clear to you, it’s probably not clear to many others. So please, do interrupt me! Also, if you would like me to explain a particular material in more depth, go over an example etc, feel free to send me an email or leave an anonymous feedback on the course webpage. Any suggestions, comments, and feedbacks are welcome and greatly appreciated.

**2 Proofs**

If you ask me what the biggest difference is between high school math and Caltech math, I would say “PROOFS”. It is the essence of higher mathematics and what makes mathematics so beautiful. But learning to write mathematical proofs can be hard.

So what does it mean to prove something? To prove something means to show something is true (that wasn’t a trick question). Every field of study has a way of “showing” that some result is true. For example, in English literature, if you wanted to show that the concept of whiteness in *Moby Dick* is tied with morality, you would write an essay that includes quotes from *Moby Dick*, other texts by Melville, and maybe from what other scholars have written about *Moby Dick*. In physics, if you wanted to show that neutrinos don’t
travel faster than the speed of light, you would write a paper citing relevant theories and the results of an
experiment designed to support your theory.

Mathematical proof is how mathematicians go about showing that something is true. In structure, a
mathematical proof is very similar to that of a short essay or a paper. You start by making a claim, and
then go about assembling a series of true statements that demonstrate that this claim is true. The only
distinction between mathematics and other fields, roughly speaking, is that the only admissible statements
in a mathematical proof are 1. things we have previously proven to be true, and 2. axioms: i.e. a collection
of statements that we have decided to assume are true. In other words, all mathematically true statements
are just consequences of a predetermined set of axioms.

Over the course of this year, we will be doing precisely that in Ma001. We will start with a set of
statements that we will “agree” to hold true, then build little by little upon those axioms until we have
beautiful theories. As such, we will start by studying various techniques of proofs such as induction and
proof by contradiction.

2.1 Words and Proofs

Let’s begin by taking a look at what you should never do in proofs:

Proof.

\[
0 < \prod_p \sin \left( \frac{\pi}{p} \right) = \prod_p \sin \left( \frac{\pi(1 + 2 \prod_{p'} p')}{p} \right) = 0
\]

This is a TERRIBLE proof because... we don’t even know what’s being proven! (FYI, this is a one-line
proof of infinitude of primes. Quite stunning in its own right but again, an awful proof).

Let’s take a look at another example of a bad proof and really analyze it:

Proof.

\[
\sqrt{xy} \leq \frac{x + y}{2} \quad (1)
\]
\[
xy \leq \frac{(x + y)^2}{4} \quad (2)
\]
\[
4xy \leq (x + y)^2 \quad (3)
\]
\[
4xy \leq x^2 + 2xy + y^2 \quad (4)
\]
\[
0 \leq x^2 - 2xy + y^2 \quad (5)
\]
\[
0 \leq (x - y)^2 \quad (6)
\]

Why is this a TERRIBLE (even worse than the first) proof? First and foremost, the proof has no words!
In fact, we have no idea what we are even proving, what \(x\) and \(y\) are supposed to be (are \(x\) and \(y\) positive,
negative, nonzero, or reals?), or what each inequality implies and leads to the conclusion we are trying to
draw. So this is the first important thing to remember about mathematical proofs: use words. Always,
explain to the reader in words what you are proving and how you are going about making said proof.

For example, the example above is an attempt at proving the arithmetic-geometric mean inequality.

**Theorem 2.1. (AM-GM Inequality)** For any two nonnegative real numbers \(x\) and \(y\), the geometric mean of
\(x\) and \(y\) is less than or equal to the arithmetic mean of \(x\) and \(y\). In other words, we have

\[
\sqrt{xy} \leq \frac{x + y}{2}
\]
With this stated, we can see the second crucial flaw in the above example. The example starts off by assuming the AM-GM inequality is true, then deduces a statement which we already know to be true. In other words, instead of proving the desired statement from something we know to be true, the example goes in the opposite direction. This does not prove the statement we are claiming. This is because the converse of a true statement need not be true: i.e. “A implies B” being true does not tell us that “B implies A” is true. Consider the following:

\[ 1 = 2 \]  
\[ 0 \cdot 1 = 0 \cdot 2 \]  
\[ 0 = 0 \]  

Clearly, 0 = 0 is a true statement. However, this does not prove the original statement, 1 = 2, which is clearly false.

Let’s try to properly prove the AM-GM inequality, starting from a true statement and then deducing the desired AM-GM as a consequence.

**Proof of Theorem 2.1.** Let \( x \) and \( y \) be nonnegative real numbers. Since the square of any real number is nonnegative, we know in particular that \( 0 \leq (x - y)^2 \). By performing some algebraic manipulation to above inequality we obtain the following:

\[ 0 \leq (x - y)^2 \]  
\[ \Rightarrow 0 \leq x^2 - 2xy + y^2 \]  
\[ \Rightarrow 4xy \leq x^2 + 2xy + y^2 \]  
\[ \Rightarrow 4xy \leq (x + y)^2 \]  
\[ \Rightarrow xy \leq \frac{(x + y)^2}{4} \]  

Since \( x \) and \( y \) are both nonnegative, \( xy \) is nonnegative. Hence, we can take the square roots of both sides to get

\[ \sqrt{xy} \leq \frac{|x + y|}{2} \]

Again, since \( x \) and \( y \) are nonnegative, so is \( x + y \). In particular, \( |x + y| = x + y \). Finally, we obtain

\[ \sqrt{xy} \leq \frac{x + y}{2} \]

which is what we wanted to prove.

\[ \Box \]

### 2.2 Avoiding Overkill

Another thing to mention in mathematics (particularly for Caltech undergrads) is to avoid overkill in proofs. Many of you have seen a lot of mathematics in high school. Consequently, when students are going through math courses, they are often tempted to use tools that they have seen in other math classes (most notoriously l’Hôpital’s rule) to attack problems. Do not do this!

There are a lot of reasons you should avoid using results not proven either by yourself in previous problem sets, in class, or in recitations. One trivial reason is that pretty much everything you will have to prove in a calculus course will have been proven elsewhere, and you won’t need to do any work at all if you were to cite any source. Another reason is that proofs involving this kind of “overkill” are usually not very illuminating. The problems that are assigned are designed to be doable with the results that you have proven so far in the course, and to broaden your understanding of the subject. As such, I strongly encourage you to use the tools that have been made available to you from the course to attack the problem.
Consider the following “overkill” for example:

**Theorem 2.2.** \( \sqrt{2} \) is irrational.

**Proof.** First, recall Fermat’s Last Theorem, a conjecture made by Fermat in 1637 and proven by Andrew Wiles in 1995:

**Theorem 2.3** (Fermat’s Last Theorem). *If \( n \) is a natural number \( \geq 3 \), the equation*

\[
a^n + b^n = c^n
\]

*has no solutions with \( a, b, c \) \( \in \mathbb{N} \).

We will now use this to prove that \( \sqrt{2} \) is irrational. We proceed by contradiction: i.e. suppose that \( \sqrt{2} \) is rational. Then, we can find \( p, q \) \( \in \mathbb{N} \) such that \( \sqrt{2} = p/q \). Then, by cubing both sides we get

\[
\frac{p^3}{q^3} = 2
\]

Multiplying both sides by \( q^3 \) we get

\[
p^3 = 2q^3 = q^3 + q^3
\]

However, Fermat’s Last Theorem says that we cannot find such natural numbers \( p \) and \( q \). Therefore, we arrive at a contradiction and hence, we conclude that \( \sqrt{2} \) is irrational. \( \square \)

To make it clear, this proof is completely valid. Yet, we don’t gain any insight into the “real” reason \( \sqrt{2} \) is not rational or what makes a number rational. Good proofs are ideally ones that illuminate the question at hand: not only do they rigorously show that the statement in question is true, they also shed light on how concepts involved in the proofs work, and how the reader might go about attacking similar problems.

### 3 Proof Techniques

In this section, we will study specific proof techniques. Let’s first set things up.

A *(mathematical) statement* or a *proposition*, is a sentence that is either true or false (but not both). An example of a statement is “1 is an integer,” which is true.

Given statements \( P \) and \( Q \), we can form derived statements. The most common are

- \( \neg P \): not \( P \)
- \( P \land Q \): \( P \) and \( Q \)
- \( P \lor Q \): \( P \) or \( Q \)
- \( P \Rightarrow Q \): \( P \) implies \( Q \)

The statement we will be trying to prove will often be either simply “\( P \) is true” or “\( P \) implies \( Q \)” i.e. we will be trying to prove either \( P \) or \( P \Rightarrow Q \). For the rest of the section, the statement we wish to prove will be “\( P \) implies \( Q \)” unless stated otherwise.

#### 3.1 Direct Proof

Of course, we can always take the most direct road. In other words we will 1. assume that \( P \) is true; then 2. use \( P \) to show that \( Q \) must be true. Let’s take a look at an example of such proof.

**Example 3.1.** Let \( m \) and \( n \) be consecutive integers. Then, show that \( m + n \) is odd.
Proof. Since \( m \) and \( n \) are consecutive integers, we can assume without loss of generality that \( n = m + 1 \). Then, we have

\[
m + n = m + (m + 1) = 2m + 1
\]

Therefore, \( m + n = 2m + 1 \) is odd.

Before we move on to different proof techniques, we should note that direct proofs are not rare at all! Very often, proofs come down to checking definitions and various axioms. Such proofs are usually direct. However, there are times when we either directly prove “\( P \) implies \( Q \)” or it may not be the easiest way to prove the desired statement.

### 3.2 Proof by Contradiction

Proof by contradiction is another technique that is often used in math. The strategy of proving by contradiction is as follows:

1. Assume that \( P \) is true.
2. Assume that \( Q \) is false i.e. \( \neg Q \) is true.
3. Using \( P \) and \( \neg Q \), deduce a contradiction.

**Example 3.2.** Prove that there are two irrational numbers \( a \) and \( b \) such that \( a^b \) is rational.

Suppose that there are no two such irrational numbers. In other words, suppose that for all irrational numbers \( a \) and \( b \), \( a^b \) is also irrational.

Let \( a = b = \sqrt{2} \). Then, by hypothesis, \( \sqrt[2]{\sqrt{2}} \) must also be irrational.

Now, let \( a = \sqrt{2} \) and \( b = \sqrt[2]{\sqrt{2}} \). Again, our hypothesis tells us that \( a^b = \sqrt[2]{\sqrt{2} \cdot \sqrt{2}} \) must be irrational. On the other hand, we have that \( a^b \) is equal to:

\[
\left(\sqrt[2]{\sqrt{2}}\right)^\sqrt{2} = \sqrt[4]{4} = \sqrt{2}^2 = 2
\]

which is clearly rational. Hence, \( a^b \) is both rational and irrational which is a contradiction.

Therefore, our initial hypothesis must be false. Hence, there are irrational numbers \( a \) and \( b \) such that \( a^b \) is rational.

### 3.3 Proof by Induction

Induction is used when we want to prove statements \( P(n) \) which depends on some variable \( n \). For example the statement \( P(n) \) could be

- The sum of the first \( n \) natural numbers is \( \frac{n(n+1)}{2} \)
- If \( q \geq 2 \), then \( q^n \geq n \)

We could check if \( P(n) \) is true for our favorite value of \( n \) using techniques we discussed before. But how could we prove that \( P(n) \) is true for all values of \( n \)? One method for doing this is called mathematical induction. Here’s how mathematical induction works.

1. (Base case) First, we show that \( P(1) \) is true.
2. (Induction step) We assume that \( P(k) \) is true for all values of \( k = 1, 2, \ldots, n \). Then, using these we prove that \( P(n + 1) \) is true.
The assumption that \( P(k) \) is true for all values \( k = 1, \ldots, n \) is called the induction hypothesis.

Let’s first start with a simple example that you might have seen in Ma 1a already.

**Example 3.3.** For all \( n \in \mathbb{N} \), show that \( S_1(n) = 1 + \cdots + n = \frac{n(n+1)}{2} \)

**Proof.** We proceed by induction on \( n \).

First, we check that \( S_1(1) = 1 \). The left hand side is just 1, and we see that the right hand side is simply \( \frac{2}{2} = 1 \). Hence, the base case is true.

Now, assume that \( S_1(k) = \frac{k(k+1)}{2} \) for all \( k = 1, \ldots, n \). We would like to show that \( S_1(n+1) = \frac{(n+1)(n+2)}{2} \).

Note that we can write \( S_1(n) = S_1(n) + (n+1) \). By induction hypothesis, we have that \( S_1(n) = \frac{n(n+1)}{2} \).

Hence, we get

\[
S_1(n+1) = S_1(n) + (n+1)
\]

\[
= \frac{n(n+1)}{2} + (n+1)
\]

\[
= \frac{n(n+1) + 2(n+1)}{2}
\]

\[
= \frac{(n+1)(n+2)}{2}
\]

Therefore, we conclude that \( S(n) = \frac{n(n+1)}{2} \) for all \( n \in \mathbb{N} \). \( \square \)

Below are a few more examples of statements that you can use induction to prove:

- For all \( n \in \mathbb{N} \), show that

\[
\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}
\]

- For all \( n \in \mathbb{N} \), show that

\[
\sum_{i=1}^{n} (-1)^i i^2 = (-1)^n \frac{n(n+1)}{2}
\]

- For all \( n \in \mathbb{N} \), show that

\[
\sum_{i=1}^{n} (2i-1) = n^2
\]

Here’s an example where we need to think a little harder.

**Example 3.4.** Prove that

\[
\begin{array}{ccccccc}
1 & 3 & 5 & 9 & 99 & 999 & 99999
\end{array}
\leq
\begin{array}{c}
\frac{1}{1000}
\end{array}
\]

**Proof.** Since \( 1000 = \sqrt{1000000} \), we might try to show that for all \( n \) we have

\[
\frac{1}{2} \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{2n}}
\]

Let’s try to prove this statement by induction. For \( n = 1 \), we have

\[
\frac{1}{2} < \frac{1}{\sqrt{2}}
\]

which is true.
So we move on to the induction step. Suppose the statement is true for \( n \). We want to show that the statement is true for \( n + 1 \) as well. i.e.

\[
\frac{1 \cdot 3 \cdots 2n - 1}{2n} \cdot \frac{2n + 1}{2n + 2} < \frac{1}{\sqrt{2n + 2}}
\]

Using the induction hypothesis, we know that the left hand side is less than \( \frac{1}{\sqrt{2n}} \cdot \frac{2n+1}{2n+3} \). Hence, it suffices to show

\[
\frac{2n + 1}{2n + 2} < \sqrt{\frac{2n}{2n + 2}}
\]

By squaring both sides, we get

\[
\frac{(2n + 1)^2}{(2n + 2)^2} < \frac{2n}{2n + 2} \Leftrightarrow (2n + 1)^2 < 2n(2n + 2)
\]

But we can easily see that above inequality is not true.

What do we do now? We might be tempted to take out our calculators and punch in the numbers (i.e. direct proof!). But we are Caltech students and we don’t give into puny problems.

Instead, we will try to prove a slightly stronger inequality. Namely, we will show that

\[
\frac{1 \cdot 3 \cdots 2n - 1}{2n} < \frac{1}{\sqrt{2n + 1}}
\]

Again, the base case is trivial so we proceed to the induction step. Suppose the inequality is true for \( n \) and let’s prove it for \( n + 1 \).

Since \((2n + 1)(2n + 3) = (2n + 2)^2 - 1 < (2n + 2)^2\), we have

\[
\frac{(2n + 1)^2}{(2n + 2)^2} < \frac{2n + 1}{2n + 3} \Rightarrow \frac{2n + 1}{2n + 2} < \sqrt{\frac{2n + 1}{2n + 3}}
\]

Using our induction hypothesis we obtain:

\[
\frac{1 \cdot 3 \cdots 2n - 1}{2n} \cdot \frac{2n + 1}{2n + 3} < \frac{1}{\sqrt{2n + 1}} \cdot \sqrt{\frac{2n + 1}{2n + 3}} = \frac{1}{\sqrt{2n + 3}}
\]

Hence, the equality holds for all \( n \in \mathbb{N} \). In particular, letting \( n = 500000 \), we see that

\[
\frac{1 \cdot 3 \cdots 999999}{1000000} < \frac{1}{\sqrt{10000001}} < \frac{1}{1000}
\]

Here’s an alternate proof showing that there always are more than one way to approach the problem.

**Proof.** Let

\[
A = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{999999}{1000000} \quad \text{and} \quad B = \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{1000000}{1000001}
\]

Clearly, \( A < B \) since \( \frac{1}{2} < \frac{2}{3}, \frac{3}{4} < \frac{4}{5} \) and so on. Hence, it follows that

\[
A^2 < AB = \frac{1}{10000001} < \frac{1}{1000000}
\]

By taking square roots of both sides, we get the desired inequality.

Let’s finish with something more fun. Before I describe the problem, I would like to say that this puzzle/riddle has been around in many forms and that this particular version is taken from io9.com with
Example 3.5. Green Eyed Dragons

Over the summer, I visited a remote island inhabited by 100 very friendly dragons all of who have green eyes. All the dragons were very hospitable and showed me around the island and told me about their history and culture. One peculiar thing that I learned is the following: If at any point, a dragon can prove that it has green eyes, it must transform into a chicken precisely at midnight on the day of the discovery (because what’s more ridiculous than a green eyed dragon?).

The dragons are not color blind but there are no mirrors so they can never see the color of their own eyes. Also, talking about eye color is something of a taboo among the dragons. So even if the dragons see that others have green eyes, they never talk about their eye color, and the dragons have been living in this blissful ignorance (as dragons).

As my stay came to a close and as I was leaving, I gathered all 100 dragons to thank them for their hospitality. Then, I decided against my better judgement and told the dragons that there is at least one green eyed dragon on the island.

Can you see what happened on the island after I left?

**Answer:** All 100 dragons turn into chickens at the midnight of the 100th day.

Let’s prove this by induction.

**Proof.** We will prove the following statement: If there are exactly \(n\) GEDs (green eyed dragons), all \(n\) dragons will simultaneously turn into chickens at the midnight of the \(n\)-th day.

The base case is simple. Suppose there is only 1 GED, call it Tom. After I leave, Tom looks around and sees no GEDs. Since there is at least 1 GED on the island, Tom realizes that I was talking about him. He curses me under his breath and turns into chicken at midnight of the first day.

Now the induction step. Suppose the statement is true up to \(n\). We wish to show that the statement is true for \(n + 1\).

Assume there are exactly \((n + 1)\) GEDs on the island. The following is what goes through all \((n + 1)\) GEDs’ heads after I leave. Let’s pick one of the GEDs, say its name is Kate. After I leave, Kate looks around and sees \(n\) GEDs (since she cannot see herself). She deduces that either 1. she doesn’t have green eyes and there are exactly \(n\) GEDs; or 2. she also has green eyes and there are exactly \((n + 1)\) GEDs. To prove that she has green eyes, she proceeds by contradiction: assume that she does not have green eyes. Hence, there must be exactly \(n\) GEDs and by induction hypothesis, all \(n\) GEDs (which Kate sees) must turn to chicken on the \(n\)-th midnight.

Come the faithful morning of the \((n + 1)\)-th day. Kate wakes up eager to find \(n\) GECs (green eyed chickens) running around. Note that since she cannot prove by the \(n\)-th midnight that she has green eyes, Kate does not turn into chicken herself. Same applies to all other \(n\) GEDs. Kate, along with \(n\) other GEDs, wakes up and finds that nothing has changed. She shudders as she finally comes to realization that a contradiction has been reached and that the initial assumption she made has to be false: she has green eyes. She, along with \(n\) other GEDs, curses me under her breath and turns into a chicken at the stroke of midnight on the \((n + 1)\)-th day.

Hence, the statement “If there are exactly \(n\) GEDs, all \(n\) GEDs will turn into GECs on the midnight of the \(n\)-th day” is true for all \(n\). In particular, when \(n = 100\) (which is our case), all 100 GEDs turn into chickens on the midnight of the 100th day.

References