Recall: Partitions of a Positive Integer

- For a positive integer $n$, we denote by $p(n)$ the number of ways to write $n$ as a sum of positive integers.
- Example. We can write $n = 5$ as
  
  $5$, $4 + 1$, $3 + 2$, $3 + 1 + 1$, $2 + 2 + 1$, $2 + 1 + 1 + 1$, $1 + 1 + 1 + 1 + 1$.

  so $p(5) = 7$.
  - $p(20) = 627$.
  - $p(100) = 190569292$. 
Recall: Ferrers Diagrams

- **Ferrers diagrams** are a graphic way of representing partitions.

\[ 14 = 6 + 4 + 3 + 1 \]

\[ p(4) = 5 \]

Hardy, Ramanujan, and Partitions

- Hardy and Ramanujan found the **approximation**
  \[ p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}. \]

- They also expressed \( p(n) \) **exactly** as an infinite sum.
The Generating Function

\[ P(x) = p(0) + p(1)x + p(2)x^2 + \cdots \]
\[ = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3) \]

Intuitive Explanation

- When opening the parentheses, \( p(4) \) is the coefficient of \( x^4 \).
Restricted Partitions #1

• Consider partitions of \( n \) with no more than \( k \) identical parts.

• For example, when \( n = 12 \) and \( k = 2 \):
  ◦ 3 + 3 + 3 + 3 and 4 + 4 + 4 are not valid.
  ◦ 5 + 5 + 2 and 2 + 2 + 4 + 4 are valid.

• **Problem.** What is the generating function of partitions that have no more than \( k \) identical parts?

Restricted Partitions #1 (cont.)

• **Special case.** Taking \( k = 1 \), we get the generating function for \( p(n \mid \text{each part is distinct}) \):

\[
1 + x + x^2 + x^3 + \cdots
\]

• What about the case of an arbitrary \( k \)?

\[
\prod_{n=1}^{\infty} \left(1 + x^n + x^{2n} + \cdots + x^{kn}\right).
\]
Restricted Partitions #2

• Consider partitions of $n$ with **only odd parts**.

• For example, when $n = 12$:
  
  $1 + 1 + 1 + \cdots + 1, 3 + 3 + 3 + 3, 11 + 1$, etc...

• **Problem.** What is the generating function of partitions with only odd parts?

\[
(1 - x)^{-1}(1 - x^3)^{-1}(1 - x^5)^{-1} \cdots = \prod_{n=1}^{\infty} (1 - x^{2n-1})^{-1}.
\]

Restricted Partitions #3

• Consider partitions of $n$ with **each part equals to at most $k$**.

• For example, when $n = 12$ and $k = 4$:
  
  $5 + 5 + 2$ and $10 + 1 + 1$ are **not valid**.

• **Problem.** What is the generating function of partitions whose parts equal to at most $k$?

\[
(1 - x)^{-1}(1 - x^2)^{-1}(1 - x^3)^{-1} \cdots (1 - x^k)^{-1} \quad = \prod_{n=1}^{k} (1 - x^n)^{-1}.
\]
**Are These the Same? #1**

- The generating function of $p(n \mid$ each part is odd) is
  \[ \prod_{n=1}^{\infty} (1 - x^{2^{n-1}})^{-1}. \]

- The generating function of $p(n \mid$ each part is even) is
  \[ \prod_{n=1}^{\infty} (1 - x^{2n})^{-1}. \]

- Does $p(n \mid$ each part is odd) = $p(n \mid$ each part is even) for every $n$?

**Answer**

- No!
  - For example, when $n$ is odd the number of even partitions is zero and the number of odd partitions is not.
Are These the Same? #2

- The generating function of \( p(n \mid \text{each part is odd}) \) is
  \[
  \prod_{n=1}^{\infty} (1 - x^{2n-1})^{-1}.
  \]

- The generating function of \( p(n \mid \text{each part is distinct}) \) is
  \[
  \prod_{n=1}^{\infty} (1 + x^n).
  \]

- Does \( p(n \mid \text{each part is odd}) = p(n \mid \text{each part is distinct}) \) for every \( n \)?

Answer

- We now prove that
  \[
  \prod_{n=1}^{\infty} (1 - x^{2n-1})^{-1} = \prod_{n=1}^{\infty} (1 + x^n).
  \]

- **Proof.** Since \( 1 + y = (1 - y^2)/(1 - y) \), we have
  \[
  \prod_{n=1}^{\infty} (1 + x^n) = \frac{\prod_{n=1}^{\infty} (1 - x^{2n})}{\prod_{n=1}^{\infty} (1 - x^n)}
  \]
  \[
  = \prod_{n=1}^{\infty} (1 - x^{2n-1})^{-1}
  \]
Odd VS Even Number of Parts

- We define
  \( e_n = p(n \mid \text{distinct parts and their # is even}) \)
  \( o_n = p(n \mid \text{distinct parts and their # is odd}) \).

- **Problem.** What is \( e_n - o_n \)?
  - When \( n = 2 \), \( e_n - o_n = 0 - 1 = -1 \).
  - When \( n = 3 \), \( e_n - o_n = 1 - 1 = 0 \).
  - When \( n = 4 \), \( e_n - o_n = 1 - 1 = 0 \).
  - When \( n = 5 \), \( e_n - o_n = 2 - 1 = 1 \).
  - When \( n = 6 \), \( e_n - o_n = 2 - 2 = 0 \).

The Generating Function

- Set \( q(n) = e_n - o_n \), and
  \[ Q(x) = 1 + q(1)x + q(2)x^2 + q(3)x^3 + \cdots \]

- **Claim.** \( Q(x) = (1 - x)(1 - x^2)(1 - x^3)\cdots \).

- **Proof.**
  - A partition \( n = s_1 + \cdots + s_k \) (where the parts are distinct), corresponds to
    \((-x^{s_1})(-x^{s_2})\cdots(-x^{s_k}).\)
  - That is, every even partition of \( n \) contributes \( x^n \) and every odd partition contributes \(-x^n\).
**The Correct Bound**

- **Theorem.** We have
  \[ e_n - o_n = \begin{cases} (-1)^m, & \text{if } n = \frac{1}{2}m(3m \pm 1), \\ 0, & \text{otherwise.} \end{cases} \]

- **Proof.**
  - Refer to partitions that are counted in \( e_n \) as *even*, and to partitions that are counted in \( o_n \) as *odd*.
  - We define a map from even to odd partitions, or from odd to even partitions, which is *almost a bijection.*

**Preparation for the Map**

- \( \lambda \) – a partition with distinct parts.
- \( s(\lambda) \) – the size of the smallest part of \( \lambda \).
- \( t(\lambda) \) – the length of the sequence that starts with the first part of \( \lambda \) and continues as long as parts decrease by 1 at each step.

\[
\begin{align*}
\begin{array}{c}
\bullet \bullet \bullet \bullet \bullet \bullet \\
\bullet \bullet \bullet \bullet \\
\bullet \bullet \\
\bullet \\
\end{array} & \quad s(\lambda) = 1 \\
\begin{array}{c}
\bullet \bullet \bullet \bullet \bullet \bullet \\
\bullet \bullet \bullet \bullet \\
\bullet \bullet \\
\bullet \\
\end{array} & \quad t(\lambda) = 3
\end{align*}
\]
The Mapping – Case 1

- If $s(\lambda) \leq t(\lambda)$, we remove the last part of $\lambda$ and add 1 to each of the first $s(\lambda)$ parts.

- This map takes an even partition to an odd partition (and vice versa).

The Mapping – Case 2

- If $s(\lambda) > t(\lambda)$, we remove 1 from each of the $t(\lambda)$ largest parts, and add a new smallest part of size $t(\lambda)$.

- This map also takes an even partition to an odd partition (and vice versa).
Examining Case 1

- When does the mapping of case 1 fails? (assuming that $s(\lambda) \leq t(\lambda)$).

- When $s(\lambda) = t(\lambda)$ = the number of parts of $\lambda$.

Case 1 is “Almost” Well Behaved

- For what values of $n$ can the problem in case 1 occur?
  - Write the number of parts as $m = s(\lambda) = t(\lambda)$.
  
  \[ n = m + (m + 1) + (m + 2) + \cdots + (2m - 1) = \frac{1}{2} m(3m - 1). \]

  - That is, when $n = \frac{1}{2} m(3m - 1)$ (for some $m \in \mathbb{N}$) case 1 fails to map one partition.
Examining Case 2

- When does the mapping of case 2 fails? (assuming that $s(\lambda) > t(\lambda)$).
- When $s(\lambda) - 1 = t(\lambda) = \text{number of parts}$.

Case 2 is “Almost” Well Behaved

- For what values of $n$ can the problem in case 2 occur?
  - The number of parts: $m = s(\lambda) - 1 = t(\lambda)$.
  - $n = (m + 1) + (m + 2) + \cdots + 2m$
  - $= \frac{1}{2}m(3m + 1)$.
- That is, when $n = \frac{1}{2}m(3m + 1)$ (for some $m \in \mathbb{N}$) case 2 fails to map one partition.
Concluding the Proof

- If \( n \neq \frac{1}{2}m(3m \pm 1) \), then the mapping is a bijection and thus \( e_n - o_n = 0 \).
  - Every even partition is taken to an odd partition.
  - Every odd partition is obtained from a unique even one.

- If \( n = \frac{1}{2}m(3m \pm 1) \), then
  - If \( m \) is even, the mapping takes the even partitions to distinct odd partitions, with one exception, so \( e_n - o_n = 1 \).
  - If \( m \) is odd, the mapping takes the odd partitions to distinct even partitions, with one exception, so \( e_n - o_n = -1 \).
A Simple Observation

- **Recall.**
  - The partitions generating function is
    \[
    P(x) = \prod_{n=1}^{\infty} (1 - x^n)^{-1}.
    \]
  - The generating function of \( e_n - o_n \) is
    \[
    Q(x) = \prod_{n=1}^{\infty} (1 - x^n).
    \]

- We thus have
  \[
  P(x) \cdot Q(x) = 1.
  \]

Consequences of the Observation

- We have \( P(x) \cdot Q(x) = 1 \). Equivalently,
  \[
  (1 + p(1)x + p(2)x^2 + \cdots)(1 - x - x^2)
  \]
Computing $p(n)$

- The above technique gives us an efficient recursive method for computing $p(n)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(n - 1)$</td>
<td>11</td>
<td>15</td>
<td>22</td>
<td>30</td>
<td>42</td>
<td>56</td>
<td>77</td>
<td>101</td>
</tr>
<tr>
<td>$p(n - 2)$</td>
<td>7</td>
<td>11</td>
<td>15</td>
<td>22</td>
<td>30</td>
<td>42</td>
<td>56</td>
<td>77</td>
</tr>
<tr>
<td>$p(n - 5)$</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>11</td>
<td>15</td>
<td>22</td>
<td>30</td>
</tr>
<tr>
<td>$p(n - 7)$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>11</td>
<td>15</td>
</tr>
<tr>
<td>$p(n - 12)$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$p(n)$</td>
<td>15</td>
<td>22</td>
<td>30</td>
<td>42</td>
<td>56</td>
<td>77</td>
<td>101</td>
<td>135</td>
</tr>
</tbody>
</table>

The End: Another Bad Joke?!?!

Three logicians walk into a bar. The bartender asks “Do you all want a beer?” The first says “I don’t know”. The second says “I don’t know”. The third says “Yes!”