Ma/CS 6a
Class 25: Partitions

Explain the significance of the following sequence: un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu...

Answer

Explain the significance of the following sequence: un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu...

These are the Catalan numbers!

(The numbers one to ten in Catalan.)
Partitions of a Positive Integer

- For a positive integer $n$, denote by $p(n)$ the number of ways to write $n$ as a sum of unordered positive integers.
- **Example.** We can write $n = 5$ as
  
  $5, \quad 4 + 1, \quad 3 + 2, \quad 3 + 1 + 1, \quad 2 + 2 + 1, \quad 2 + 1 + 1 + 1, \quad 1 + 1 + 1 + 1 + 1.$

  so $p(5) = 7$.
  - $p(20) = 627$.
  - $p(100) = 190569292$.

Ferrers Diagrams

- **Ferrers diagrams** are a graphic way of representing partitions.

  $14 = 6 + 4 + 3 + 1$

  $p(4) = 5$
A Simple Observation

- **Claim.** Let \( n \) and \( r \) be positive integers. Then

\[
p(n \mid \text{number of parts } \leq r) = p(n + r \mid \text{number of parts } = r).
\]

- **Proof.** We find a **bijection** between the two sets of partitions:

\[
\begin{align*}
n + r & \longleftrightarrow n \\
\end{align*}
\]
Detailed Proof

- We describe a bijection between the sets:
  - $P_n$ - Partitions of $n$ with at most $r$ parts.
  - $P_{n+r}$ - Partitions of $n + r$ with exactly $r$ parts.
- Given a partition of $P_n$, we add a new first column with $r$ elements, obtaining a partition of $P_{n+r}$.
- Given a partition of $P_{n+r}$, we remove the first column to obtain a partition of $P_n$.

Conjugate Partitions

- Two partitions of a number $n$ are said to be conjugate if one is obtained from the other by switching the rows and columns in the Ferrers Diagram.
Using Conjugate Partitions

- Consider a pair of conjugate partitions \( \alpha, \beta \). The **size of the largest part of \( \alpha \)** is the **number of parts of \( \beta \)**.
- Using a **bijection** argument as before, we have

\[
 p(n \mid \text{largest part of size } m) = p(n \mid \text{number of parts } = m).
\]

Self-Conjugation

- A partition is **self-conjugate** if it is its own conjugate.
- **Claim.**

\[
 p(n \mid \text{self–conjugate}) = p(n \mid \text{the parts are distinct and odd}).
\]
Self Conjugation Proof

\[ p(n \mid \text{self–conjugate}) = p(n \mid \text{the parts are distinct and odd}). \]

\[ \text{Proof.} \quad \text{As before, we find a bijection between the two sets of partitions.} \]

\[ \text{Given a self conjugate partition, let } k_i \text{ be the number of elements in the } 1^{\text{st}} \text{ row and column after removing the first } i - 1 \text{ rows and columns. For } i < j, \text{ we have } k_i > k_j. \]

\[ \text{We use the } 2k_i - 1 \text{ elements in the } i^{\text{'th}} \text{ “row and column” to create the } i^{\text{'th}} \text{ row.} \]

Partitions and Generating Functions

- To calculate \( p(i) \), we define a generating function for the number of partitions:
  \[ P(x) = p(0) + p(1)x + p(2)x^2 + \cdots \]
  \[ \text{By convention, we write } p(0) = 1. \]

- We have as many initial values as we like:
  \[ p(1) = 1, \quad p(2) = 2, \quad p(3) = 3, \quad p(4) = 5, \quad p(5) = 7, \ldots \]

- Not clear how to find a recursive relation.
Warm-Up Question

- **Recall.** For any positive integer $n$, we have
  \[(1 - x^n)^{-1} = 1 + x^n + x^{2n} + x^{3n} + \cdots\]

- $p_n(m) = \text{number of partitions of } m$
  where each part is of size $n$.
  \[
p_n(m) = \begin{cases} 
1, & \text{if } n|m, \\
0, & \text{otherwise}.
\end{cases}
\]

- The corresponding generating function:
  \[P_n(x) = p_n(0) + p_n(1)x + p_n(2)x^2 + \cdots
  = 1 + x^n + x^{2n} + x^{3n} + \cdots
  = (1 - x^n)^{-1}.
\]

A Bit of Progress

- Let $p_{n,m}(i)$ denote the number of partitions of $i$
  where each part is equal to either $m$ or $n$.

- Let
  \[P_{n,m}(x) = p_{n,m}(0) + p_{n,m}(1)x + p_{n,m}(2)x^2 + \cdots
  = (1 + x^n + x^{2n} + \cdots)(1 + x^m + x^{2m} + \cdots)
  = (1 - x^n)^{-1}(1 - x^m)^{-1}.
\]
Intuitive Explanation

- When opening the parentheses, $p_{2,4}(10)$ is the coefficient of $x^{10}$.

\[
(1 + x^2 + x^4 + x^6 + x^8 + \cdots)(1 + x^4 + x^8 + \cdots)
\]

\[
2 + 4 + 4
\]

\[
(1 + x^2 + x^4 + x^6 + x^8 + \cdots)(1 + x^4 + x^8 + \cdots)
\]

\[
2 + 2 + 2 + 4
\]

\[
(1 + x^2 + x^4 + x^6 + x^8 + x^{10} + \cdots)(1 + x^4 + \cdots)
\]

\[
2 + 2 + 2 + 2 + 2
\]

Hardy, Ramanujan, and Partitions

- Hardy and Ramanujan found the approximation

\[
p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}.
\]

- They also expressed $p(n)$ exactly as an infinite sum.
Changing a Dollar

- **Problem.** In how many ways can a dollar be exchanged for quarters (25c), dimes (10c), and nickels (5c)?
- To make the numbers simpler, we can divide everything by 5:
  - In how many ways can we write 20 as a sum of 1’s, 2’s, and 5’s.
  - The coefficient of $y^{20}$ in $(1 - y)^{-1}(1 - y^2)^{-1}(1 - y^5)^{-1}$.

Number Crunching

- First, let us calculate
  \[(1 - y^2)^{-1}(1 - y^5)^{-1} = (1 + y^2 + y^4 + \cdots + y^{20})(1 + y^5 + y^{10} + y^{15})\]
Number Crunching (cont.)

- We have
  \[(1 - x^2)^{-1}(1 - x^5)^{-1}\]
  \[= 1 + y^2 + y^4 + y^5 + y^6 + y^7 + y^8 + y^9 + 2y^{10} + y^{11} + 2y^{12} + y^{13} + 2y^{14} + 2y^{15} + 2y^{16} + 2y^{17} + 2y^{18} + 2y^{19} + 3y^{20}.\]

- What is the coefficient of \(y^{20}\) in \[(1 - y)^{-1}(1 - y^2)^{-1}(1 - y^5)^{-1}\]?
  
  ◦ Every element of \((1 - y^2)^{-1}(1 - y^5)^{-1}\) corresponds to one way of writing 20:
  \[1 + 1 + 1 + 1 + 1 + 1 + 1 + 2 + 1 + 2 + 1 + 2 + 2 + 2 + 2 + 2 + 3 = 29\]

Back to General Partitions

- **Theorem.** The generating function of \(p(n)\) can be written as
  \[P(x) = p(0) + p(1)x + p(2)x^2 + \cdots\]
  \[= \prod_{i=1}^{\infty}(1 - x^i)^{-1}\]
  \[= (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3)\]
Proof Sketch

- We need to verify that the coefficient of $x^n$ in $P(x)$ is $p(n)$.
  - Consider a partition $n = m_1 s_1 + m_2 s_2 + \ldots + m_k s_k$, where $s_1, \ldots, s_k$ are distinct numbers and $m_i$ is the number of parts of size $s_i$ in the partition.
    - In $\prod_{i=1}^{\infty} (1 - x^i)^{-1}$, this partition corresponds to taking $x^{m_i s_i}$ from \((1 + x^{s_i} + x^{2s_i} + \ldots)\).
    - Similarly, any choice of elements from the parentheses in $\prod_{i=1}^{\infty} (1 - x^i)^{-1}$ that yields $x^n$ corresponds to a partition of $n$.

A Small Issue

- Our proof is fine if we have a product of finitely many terms, but in $\prod_{i=1}^{\infty} (1 - x^i)^{-1}$ we have products of infinitely many terms!
  - When proving that the coefficient of $x^n$ is $p(n)$, it suffices to consider $\prod_{i=1}^{n} (1 - x^i)^{-1}$.
Restricted Partitions #1

• Consider partitions of $n$ with no more than $k$ identical parts.

• For example, when $n = 12$ and $k = 2$:
  ◦ $3 + 3 + 3 + 3$ and $4 + 4 + 4$ are not valid.
  ◦ $5 + 5 + 2$ and $2 + 2 + 4 + 4$ are valid.

• Problem. What is the generating function of partitions that have no more than $k$ identical parts?

Restricted Partitions #1 (cont.)

• Special case. Taking $k = 1$, we get the generating function for $p(n \mid$ each part is distinct): 
  \[(1 + x)(1 + x^2)(1 + x^3) \cdots\]

• What about the case of an arbitrary $k$? 
  \[
  \prod_{n=1}^{\infty} \left(1 + x^n + x^{2n} + \cdots + x^{kn}\right).
  \]
Restricted Partitions #2

• Consider partitions of \( n \) with **only odd parts**.

• For example, when \( n = 12 \):
  \[
  1 + 1 + 1 + \cdots + 1, 3 + 3 + 3 + 3, 11 + 1, \text{ etc...}
  \]

• **Problem.** What is the generating function of partitions with only odd parts?
  \[
  (1 - x)^{-1} (1 - x^3)^{-1} (1 - x^5)^{-1} \cdots \]
  \[
  = \prod_{n=1}^{\infty} (1 - x^{2n-1})^{-1}.
  \]

Restricted Partitions #3

• Consider partitions of \( n \) with **only even parts**.

• For example, when \( n = 12 \):
  \[
  10 + 2, 2 + 2 + \cdots + 2, 4 + 4 + 4, \text{ etc...}
  \]

• **Problem.** What is the generating function of partitions with only even parts?
  \[
  (1 - x^2)^{-1} (1 - x^4)^{-1} (1 - x^6)^{-1} \cdots \]
  \[
  = \prod_{n=1}^{\infty} (1 - x^{2n})^{-1}
  \]
Restricted Partitions #4

- Consider partitions of $n$ with each part equals to at most $k$.
- For example, when $n = 12$ and $k = 4$:
  - $5 + 5 + 2$ and $10 + 1 + 1$ are not valid.
- **Problem.** What is the generating function of partitions whose parts equal to at most $k$?

$$(1 - x)^{-1}(1 - x^2)^{-1}(1 - x^3)^{-1} \cdots (1 - x^k)^{-1}$$

$$= \prod_{n=1}^{k} (1 - x^n)^{-1}. $$

The End: Teaching Survey