Reminder: Generating Functions

- Given an infinite sequence of numbers $a_0, a_1, a_2, \ldots$, the **generating function** of the sequence is the power series
  
  $a_0 + a_1 x + a_2 x^2 + \cdots$

- **Example.**
  
  ◦ Recall the **Fibonacci numbers**:
    
    $F_0 = F_1 = 1 \quad F_i = F_{i-1} + F_{i-2}$.
  
  ◦ The corresponding generating function is
    
    $1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$
Reminder: Using Generating Functions

• By rephrasing the solution to a problem as a generating function, we obtain a recursion relation such as
  \[ A(x) = x + 5xA(x) - 6x^2A(x). \]

• Solving the problem is equivalent to finding the power series representation of \( A(x) \).

Homogeneous Linear Recursion

• The generating function of the Fibonacci numbers \( A(x) = a_0 + a_1x + a_2x + \cdots \) satisfies
  \[ a_0 = a_1 = 1, \quad a_i = a_{i-1} + a_{i-2} \text{ (for } i \geq 2). \]

• This is a special case of the homogeneous linear recursion (or HLR) defined as
  \[ a_0 = c_0, a_1 = c_1, \ldots, a_{k-1} = c_{k-1}, \] \[ a_{n+k} + d_1a_{n+k-1} + \cdots + d_ka_n = 0. \]
HLR Property

- **Theorem.** Given a generating function \( A(x) \) with an HLR recursion
  \[ a_{n+k} + d_1 a_{n+k-1} + \cdots + d_k a_n = 0, \]
  we have
  \[ A(x) = \frac{R(x)}{1 + d_1 x + \cdots + d_k x^k}, \]
  where \( R(x) \) is a polynomial and \( \deg R(x) < k \).

Example: Fibonacci Numbers

- The generating function of the **Fibonacci numbers** \( A(x) = a_0 + a_1 x + a_2 x + \cdots \)
  satisfies
  - \( a_0 = a_1 = 1 \).
  - \( a_i - a_{i-1} - a_{i-2} = 0 \) (for \( i \geq 2 \)).

\[ A(x) = 1 + x + x^2 (a_0 + a_1) + x^3 (a_1 + a_2) + \cdots \]
\[ = 1 + x^2 (a_0 + a_1 x + a_2 x^2 + \cdots) \]
\[ + x (a_0 + a_1 x + a_2 x^2 + \cdots) \]
\[ = 1 + xA(x) + x^2 A(x). \]
Fibonacci Numbers (cont.)

- We have
  \[ A(x) = 1 + xA(x) + x^2 A(x). \]
- That is,
  \[
  A(x) = \frac{1}{1 - x - x^2}.
  \]

Proof of HLR Property

- We rewrite
  \[
  A(x) = \frac{R(x)}{1 + d_1 x + \cdots + d_k x^k}
  \]
as
  \[
  R(x) = (1 + d_1 x + \cdots + d_k x^k)A(x)
  = (1 + d_1 x + \cdots + d_k x^k)(a_0 + a_1 x + a_2 x^2)\]
The Auxiliary Equation

- The **auxiliary equation** of the HLR
  
  \[
  a_0 = c_0, a_1 = c_1, \ldots, a_{k-1} = c_{k-1}. \\
  a_{n+k} + d_1 a_{n+k-1} + \cdots + d_k a_n = 0
  \]
  
  is
  
  \[
  t^k + d_1 t^{k-1} + \cdots + d_k = 0.
  \]

  - If the auxiliary equation has **k roots** (not necessarily distinct), then we can rewrite it as
  
  \[
  (t - \alpha_1)^{m_1} (t - \alpha_2)^{m_2} \cdots (t - \alpha_s)^{m_s} = 0,
  \]
  
  where \( m_1 + \cdots + m_s = k \).

Stronger HLR Property

- **Theorem.** Consider the sequence
  
  \( a_0, a_1, \ldots \), that is defined by an HLR with auxiliary equation
  
  \[
  (t - \alpha_1)^{m_1} (t - \alpha_2)^{m_2} \cdots (t - \alpha_s)^{m_s} = 0.
  \]
  
  Then
  
  \[
  a_n = P_1(n)\alpha_1^n + P_2(n)\alpha_2^n + \cdots + P_s(n)\alpha_s^n,
  \]
  
  where \( P_i(n) \) is a polynomial of degree at most \( m_i - 1 \).
Using the Stronger HLR Property

• **Problem.** Solve the following HLR.

\[ u_0 = 0, \quad u_1 = -9, \quad u_2 = -1, \quad u_3 = 21. \]

\[ u_{n+4} - 5u_{n+3} + 6u_{n+2} + 4u_{n+1} - 8u_n = 0. \]

• **Solution.**

  ◦ The auxiliary equation is

\[ t^4 - 5t^3 + 6t^2 + 4t - 8 = 0. \]

  ◦ This can be rewritten as

\[ (t - 2)^3(t + 1) = 0. \]

  ◦ By the theorem, we have

\[ u_n = P_1(n)\alpha_1^n + P_2(n)\alpha_2^n \]

\[ = P_1(n)2^n + P_2(n)(-1)^n. \]

Solution (cont.)

• From the auxiliary equation

\[ (t - 2)^3(t + 1) = 0, \]

we know that

\[ u_n = (An^2 + Bn + C)2^n + D(-1)^n. \]

• From the initial values

\[ u_0 = 0, \quad u_1 = -9, \quad u_2 = -1, \quad u_3 = 21, \]

we get the system of equations

\[ C + D = 0, \]

\[ 2A + 2B + 2C - D = -9, \]

\[ 16A + 8B + 4C + D = -1, \]

\[ 72A + 24B + 8C - D = 21. \]
Concluding the Solution

• We have
  \[u_n = (An^2 + Bn + C)2^n + D(-1)^n.\]

• From the initial conditions, we obtain
  \[C + D = 0,\]
  \[2A + 2B + 2C - D = -9,\]
  \[16A + 8B + 4C + D = -1,\]
  \[72A + 24B + 8C - D = 21.\]

• Solving these equations yield \(A = 1,\)
  \(B = -1,\)
  \(C = -3,\)
  \(D = 3.\)
  Therefore
  \[u_n = (n^2 - n - 3)2^n + 3(-1)^n.\]

Pineapples Like Fibonacci Numbers!

(the number of strips of each of the three types)
The Catalan Numbers

- **The Catalan numbers**. An extremely useful sequence of numbers.
- In the exercises of the book “Enumerative Combinatorics” by Stanley, there are over 150 problems whose solution is the Catalan numbers.
- Obtained by Euler and Lamé (not Catalan!)

Convex Polygons

- A polygon is **convex** if no line segment between two of its vertices intersects the outside of the polygon.
- Equivalently, every interior angle of a convex polygon is smaller than 180°.
Triangulating of a convex Polygon

- A triangulation of a convex polygon $P$ is the addition of non-crossing diagonals of $P$, partitioning the interior of $P$ into triangles.

Number of Triangulations

- Let $c_n$ denote the number of different triangulations of a convex polygon with $n + 2$ vertices.
  - We set $c_0 = 1$.
  - $c_1 = 1$.
  - $c_2 = 2$.
  - $c_3 = 5$.
  - $c_4 = 14$. 
A Recursive Relation

- We have **initial values** for \( c_n \). Now we need a **recursive relation**.
  - Consider a **side ab** of an \( n \)-sided convex polygon \( P \).
  - In every triangulation, \( ab \) belongs to exactly one triangle \( \Delta \).
  - The third vertex of \( \Delta \) can be any of the other \( n - 2 \) vertices of \( P \).

A Recursive Relation (cont.)

- The number of triangulations that contain the triangle \( abv_4 \) is \( c_2 c_3 \).
- The number of triangulations that contain the triangle \( abv_5 \) is \( c_1 c_4 \).
- Recursive relation:
  \[
  c_{n-2} = \sum_{i=0}^{n-3} c_i c_{n-3-i}
  \]
Are the Catalans an HLR?

- We have the initial values
  - $c_0 = 1, c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 14$.
- We have the relation
  $$c_{n-2} = \sum_{i=0}^{n-3} c_i c_{n-3-i}.$$

- Is this an HLR?
  - No! This is not linear and number of elements in the recursion changes according to $n$.

Solving the Recurrence

- We have the generating series
  $$C(x) = c_0 + c_1 x + c_2 x^2 + \ldots$$
  - We consider
    $$C(x)^2 = c_0 c_0 + (c_0 c_1 + c_1 c_0) x + (c_0 c_2 + c_1 c_1 + c_2 c_0) x^2 + \ldots$$
  - Writing $C(x)^2 = \sum_n d_n x^n$, we get
    $$d_n = \sum_{i=0}^{n} c_i c_{n-i},$$
    we have
    $$C(x)^2 = 1 + c_2 x + c_3 x^2 + \ldots.$$
Solving the Recursion (cont.)

- We have
  \[ C(x)^2 = 1 + c_2 x + c_3 x^2 + \ldots. \]

- That is,
  \[ C(x) = 1 + x C(x)^2. \]

Solving the Recursion (cont.)

- We have \( C(x) = 1 + x C(x)^2. \)
- Setting \( y = C(x) \), we obtain the quadratic equation
  \[ xy^2 - y + 1 = 0, \]
  or
  \[ C(x) = y = \frac{1 \pm \sqrt{1 - 4x}}{2x}. \]

- How can we handle \( \sqrt{1 - 4x} \)?
More Binomial Formulas

- Recall that

\[(x + 1)^n = \sum_{i \geq 0} \binom{n}{i} x^i.\]

- By defining \(\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}\) also for negative \(m\)'s, this formula is generalized also to negative powers.

- We can also consider values of \(m\) that are not integers!

- The binomial formula holds for fractional \(n\) (we will not prove this).

Fractional Powers

- Using the fractional formula, we have

\[\left(1 - 4x\right)^{1/2} = \sum_{i=0}^{\infty} \binom{1/2}{i} (-4x)^i\]

\[= 1 + \frac{1/2}{1!} \cdot (-4x) + \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{16x^2}{2!} + \cdots\]

- This implies

\[C(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x} = \frac{1 \pm \sum_{i=0}^{\infty} \binom{1/2}{i} (-4x)^i}{2x}.\]
Two Solutions

\[ C(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x} = \frac{1 \pm \sum_{i=0}^{\infty} \left(\frac{1/2}{i}\right) (-4x)^i}{2x}. \]

• Considering the plus and minus cases separately, we have

\[ C_-(x) = \frac{-1}{2} \sum_{i=1}^{\infty} \binom{1/2}{i} (-4)^i x^{i-1}. \]

\[ C_+(x) = \frac{1}{x} + \frac{1}{2} \sum_{i=1}^{\infty} \binom{1/2}{i} (-4)^i x^i. \]

The Correct Solution

\[ C_-(x) = \frac{-1}{2} \sum_{n=1}^{\infty} \binom{1/2}{n} (-4)^n x^{n-1}. \]

\[ C_+(x) = \frac{1}{x} + \frac{1}{2} \sum_{n=1}^{\infty} \binom{1/2}{n} (-4)^n x^n. \]

• Which one is the correct solution?

◦ We saw that \(x^{-1}\) is not well defined!

\[ c_n = \frac{-1}{2} \binom{1/2}{n+1} (-4)^{n+1} = \frac{(-4)^{n+1}}{2(n+1)!} \cdot \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdots \frac{-(2n-1)}{2}. \]
Tidying Up

\[ c_n = \frac{(-4)^{n+1}}{2(n+1)!} \cdot \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdots \frac{-(2n-1)}{2} \]

\[ = \frac{2^n}{(n+1)!} \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1) \]

\[ = \frac{2^n \cdot (2n)!}{(n+1)! \cdot n! \cdot 2^n} \]

\[ = \frac{1}{n+1} \binom{2n}{n}. \]

This is the \( n \)'th Catalan number!

Happy Thanksgiving!