Ma/CS 6a
Class 22: Power Series

Power Series

- **Monomial**: $ax^i$.
- **Polynomial**:
  \[ a_0 + a_1x + a_2x^2 + \cdots + a_nx^n. \]
- **Power series**:
  \[ A(x) = a_0 + a_1x + a_2x^2 + \cdots \]

- Also called **formal power series**, because we do not think about the meaning of $x$. 
Sums and Products

- We define sums and products of power series as in the case of polynomials.
  - \( A(x) = a_0 + a_1 x + a_2 x^2 + \cdots \)
  - \( B(x) = b_0 + b_1 x + b_2 x^2 + \cdots \)

\[
A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots
\]

\[
A(x)B(x) = (a_0 b_0) + (a_1 b_0 + a_0 b_1)x + (a_2 b_0 + a_1 b_1 + a_0 b_2)x^2 + \cdots
\]

More Sums and Products

- We define sums and products of power series as in the case of polynomials.
  - \( A(x) = a_0 + a_1 x + a_2 x^2 + \cdots \)
  - \( B(x) = b_0 + b_1 x + b_2 x^2 + \cdots \)
  - \( C(x) = A(x) + B(x) \).
  - \( D(x) = A(x)B(x) \).
  - \( c_i = a_i + b_i \).
  - \( d_i = \sum_{j=0}^{i} a_j b_{i-j} \).
Commutativity

• A group $G$ is said to be **commutative** or **Abelian** if every $a, b \in G$ satisfy $ab = ba$.

• **Commutative or not?**
  ◦ Integers under addition ✔
  ◦ $n \times n$ matrices under addition. ✔
  ◦ $S_n$ under composition of permutations. ✗
  ◦ Symmetries of the square. ✗

Rings

• A **ring** is a set $R$ together with **two binary operations** $+$ and $\cdot$ that satisfy
  ◦ The set $R$ is a **commutative group** under $+$.
  ◦ The operation $\cdot$ satisfies the closure, associativity, and identity properties.
  ◦ **Distributive laws**. For any $a, b, c \in R$, we have
    $$a(b + c) = ab + ac,$$
    $$(a + b)c = ac + bc.$$
Is this a Ring?

- The set \( \mathbb{Z}_4 = \{0,1,2,3\} \) under addition and multiplication mod 4.
  - We already know that \( \mathbb{Z}_4 \) under addition mod 4 is a commutative group.
  - For multiplication, we have closure, associativity, and identity.
  - Standard addition and multiplication are distributive, so the same holds under mod 4.

Is this a Ring? #2

- \( 2 \times 2 \) matrices with real entries, under matrix addition and multiplication.
  - \( 2 \times 2 \) matrices with real entries under addition form a commutative group.
  - For multiplication, we have closure, associativity, and identity.
  - Matrix addition and multiplication are distributive.
Polynomial Ring

- The polynomial ring $\mathbb{R}[x]$ is the set of polynomials in $x$ with coefficients in $\mathbb{R}$ and standard addition and multiplication.
- The set of polynomials under addition is a group.
- Properties of multiplication:
  - Closure. The product of two polynomials in $\mathbb{R}[x]$ is a polynomial in $\mathbb{R}[x]$.
  - Associativity. By the associativity of the standard multiplication.
  - Identity. We have $1 \in \mathbb{R}[x]$.
- Standard addition and multiplication are distributive.

Ring of Power Series

- The power series ring $\mathbb{R}[[x]]$ is the set of power series in $x$ with coefficients in $\mathbb{R}$ under addition and multiplication.
- The set of power series under addition is a group.
- Properties of $\cdot$:
  - Closure. By our definition, the product of two power series is a power series.
  - Associativity. Our definition of multiplication is associative.
  - Identity. We have $1 \in \mathbb{R}[[x]]$.
- Distributivity is not hard to verify.
What is Missing?

- In $\mathbb{R}[x]$ and $\mathbb{R}[[x]]$, why don’t we have groups with respect to the operation $\cdot$?
  - A multiplicative inverse does not always exist!
  - In $\mathbb{R}[x]$ only constant polynomials have an inverse.
  - What about $\mathbb{R}[[x]]$?
  - Is there an inverse of $1 - x$?
    
    $(1 - x)A(x) = 1.$
    
    $A(x) = 1 + x + x^2 + x^3 + \ldots$

Inverses in a Power Series

- **Theorem.** A power series
  
  $A(x) = a_0 + a_1x + a_2x^2 + \cdots \in \mathbb{R}[[x]]$
  
  has an inverse if and only if $a_0 \neq 0$.

- **Proof.** First assume that $A(x)$ has an inverse $B(x)$.
  
  $(a_0 + a_1x + a_2x^2 + \cdots)(b_0 + b_1x + b_2x^2$
Proof Cont.

- **Proof (cont.). Assume that** $a_0 \neq 0$.
  - We need to solve
    
    \[
    \begin{align*}
    a_0 b_0 &= 1, \\
    a_1 b_0 + a_0 b_1 &= 0, \\
    a_2 b_0 + a_1 b_1 + a_0 b_2 &= 0, \\
    &\vdots
    \end{align*}
    \]
  - Since $a_0 \neq 0$, we obtain the solution
    
    \[
    \begin{align*}
    b_0 &= a_0^{-1}, \\
    b_1 &= -a_1 b_0 a_0^{-1}, \\
    b_2 &= (a_1 b_1 + a_2 b_0) a_0^{-1} \\
    &\vdots
    \end{align*}
    \]

More on Inverses

- **Notation.**
  - The inverse of $A(x)$ is sometimes written as $A(x)^{-1}$ or as $\frac{1}{A(x)}$.

- For example, we have
  
  \[
  \frac{1 + x}{1 - x} = (1 + x)(1 - x)^{-1} = (1 + x)(1 + x + x^2 + \cdots) = 1 + 2x + 2x^2 + 2x^3 + \cdots
  \]
Recall: The Binomial Theorem

• By the **binomial theorem**, for \( n \geq 1 \) we have

\[
(x + 1)^n = \sum_{0 \leq i, j \leq n} \binom{n}{i} x^i 1^j
\]

\[
= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} + \cdots + \binom{n}{n}
\]

• What about the case where \( n \) is negative?

Negative Powers

• What is \((x + 1)^{-1}\)?

\[
(x + 1)(a_0 + a_1 x + a_2 x^2 + \cdots) = 1,
\]

\[
(x + 1)^{-1} = 1 - x + x^2 - x^3 + \cdots
\]

• What is \((x + 1)^{-2}\)?

\[
(x + 1)^{-2} = (\frac{1}{x + 1})^2
\]

\[
= (1 - x + x^2 - x^3 + \cdots)(1 - x + x^2 - x^3)
\]
Negative Powers Formula

- **Theorem.** For any positive integer $m$,

\[
(x + 1)^{-m} = \sum_{n \geq 0} (-1)^n \binom{m + n - 1}{n} x^n.
\]

- **Examples.**
  
  - \((x + 1)^{-1} = \sum_{n \geq 0} (-1)^n x^n = 1 - x + x^2 - x^3 + \ldots\)
  
  - \((x + 1)^{-2} = \sum_{n \geq 0} (-1)^n (n + 1) x^n = 1 - 2x + 3x^2 - 4x^3 + \ldots\)

**Proof**

- We look for the coefficient of \(x^k\) in

\[
(x + 1)^{-m} = \left((x + 1)^{-1}\right)^m = \left(\sum_{n \geq 0} (-1)^n x^n\right)^m.
\]

- To simplify, we replace \(y = -x\) and have

\[
(1 - y)^{-m} = \left((1 - y)^{-1}\right)^m = \left(\sum_{n \geq 0} y^n\right)^m.
\]

- The problem turns to: **In how many ways can we write \(k\) as a sum of \(m\) non-negative integers.**
Proof (cont.)

In how many ways can we write \( k \) as a sum of \( m \) non-negative integers?

- This is equivalent to choosing \( m - 1 \) cells in an array of \( m + k - 1 \) cells.
- There are \( \binom{m + k - 1}{m - 1} = \binom{m + k - 1}{k} \) ways.

\[
\begin{array}{cccccc}
\times & \times & \times & \times & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}
\]

1 or \( y \)
3 or \( y^3 \)
0 or \( y^0 \)

Concluding the Proof

- There are \( \binom{m + k - 1}{k} \) ways to write \( k \) as a sum of at most \( m \) positive integers.
- **The coefficient of** \( y^k \) **in**

\[
(1 - y)^{-m} = \left((1 - y)^{-1}\right)^m = \left(\sum_{n \geq 0} y^n\right)^m
\]

is \( \binom{m + k - 1}{k} \). This implies

\[
(x + 1)^{-m} = \sum_{k \geq 0} (-1)^k \binom{m + k - 1}{k} x^k.
\]
Combining Both Cases?

- For integers $m \geq 1$ we have
  \[(x + 1)^m = \sum_{0 \leq n \leq m} \binom{m}{n} x^n.\]

- For integers $m \leq -1$ we have
  \[(x + 1)^m = \sum_{n \geq 0} (-1)^n \binom{-m + n - 1}{n} x^n.\]

- Can we combine the two formulas into one?

Generalizing the Binomial Coefficients

- Given an integer $m$ and a positive integer $n$, we define
  \[
  \binom{m}{0} = 1 \quad \text{and} \quad \binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}.
  \]

- This definition subsumes the standard binomial coefficients $\binom{m}{n}$ (with $m, n \geq 0$).
  - If $n \leq m$, this is identical to $\frac{m!}{n!(m-n)!}$.
  - If $n > m$, we have $\binom{m}{n} = 0$. 
Negative Binomial Numbers

• Given positive integers $m$ and $n$, we have

$$\binom{-m}{n} = \frac{-m(-m-1)\ldots(-m-n+1)}{n!}$$

$$= (-1)^n \frac{m(m+1)\ldots(m+n-1)}{n!}$$

$$= (-1)^n \binom{m+n-1}{n}.$$

Combining Both Cases

• For integers $m \geq 1$ we have

$$(x + 1)^m = \sum_{0 \leq n \leq m} \binom{m}{n} x^n.$$

• For integers $m \leq -1$ we have

$$(x + 1)^m = \sum_{n \geq 0} (-1)^n \binom{-m + n - 1}{n} x^n.$$

• Either way, we have

$$(x + 1)^m = \sum_{n \geq 0} \binom{m}{n} x^n.$$. 
A Variant

- **Problem.** Find the value of \((1 + ax)^m\) for any integer \(m\) and \(a \in \mathbb{R}\).

- **Solution.**
  - Substitute \(y = ax\). We have
    \[
    (1 + y)^m = \sum_{n \geq 0} \binom{m}{n} y^n.
    \]
  - By bringing \(x\) back, we have
    \[
    (1 + ax)^m = \sum_{n \geq 0} \binom{m}{n} a^n x^n.
    \]

An Example

- **Problem.** Write the power series of
  \[
  \frac{2 + x}{1 - 3x + 3x^2 - x^3}.
  \]

- **Solution.**
  - First, \((1 - 3x + 3x^2 - x^3) = (1 - x)^3\).
  - By the previous theorem, we have
    \[
    (1 - x)^{-3} = \sum_{n \geq 0} \binom{-3}{n} (-x)^n = \sum_{n \geq 0} \binom{n + 2}{n} x^n.
    \]
  
    \[
    (2 + x)(1 - x)^{-3} = 2 + \sum_{n \geq 1} \left(2 \binom{n + 2}{n} + \binom{n + 1}{n - 1}\right) x^n.
    \]
The End

(x, why?)

Say, Cube, do you consider yourself to be religious?

I believe there are higher powers than us, if that's what you mean.

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