The 15 Puzzle

- **Problem.** Start with the configuration on the left and move the tiles to obtain the configuration on the right.
The 15 Puzzle (cont.)

- The game became a craze in the U.S. in 1880.
- **Sam Loyd**, a famous chess player and puzzle composer, offered a $1,000 prize for anyone who could provide a solution.

Reminder: Permutations

- **Problem.** Given a set \{1,2, \ldots, n\}, in how many ways can we order it?
- **The general case.**
  \[
  n! = n \cdot (n - 1) \cdot \ldots \cdot 2 \cdot 1
  \]

Options for placing 1
Options for placing 2
Options for placing $n$
The 15 Puzzle and Permutations

- How can a configuration of the 15-puzzle be described as a permutation?
  - Denote the missing tile as 16.
  - The board below corresponds to the permutation 1 16 3 4 6 2 11 10 5 8 7 9 14 12 15 13.

Permutations as Functions

- We can consider a permutation as a bijection from the set \( \{1, 2, \ldots, n\} \) to itself.
  
  \[
  \begin{array}{ccccccc}
  1 & 2 & 3 & 4 & 5 & 6 \\
  \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  5 & 1 & 3 & 2 & 6 & 4 \\
  \end{array}
  \]

- Denote the bijection as \( \alpha \).
  - \( \alpha(1) = 5 \).
  - \( \alpha(3) = 3 \).
The Permutation Set $S_n$

- $S_n$ – The set of permutations of $\mathbb{N}_n = \{1, 2, 3, \ldots, n\}$.
- We have $|S_n| = n!$.
- The set $S_3$:

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Composition of Permutations

- $\alpha(1) = 2, \quad \alpha(2) = 4, \quad \alpha(3) = 5, \quad \alpha(4) = 1, \quad \alpha(5) = 3$.
- $\beta(1) = 3, \quad \beta(2) = 5, \quad \beta(3) = 1, \quad \beta(4) = 4, \quad \beta(5) = 2$.
- What is the function $\beta\alpha$?

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<td>$\beta$:</td>
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<td>5</td>
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</table>

First apply $\alpha$ and then $\beta$. 

Closure of $S_n$

- **Claim.** If $\alpha$ and $\beta$ are in $S_n$, so is $\alpha \beta$.
- By definition, $\alpha \beta$ is a function from $\mathbb{N}_n$ to itself.
- It remains to show that for every $i \in \mathbb{N}_n$ there is a unique $j \in \mathbb{N}_n$ such that $i = \alpha \beta (j)$.
  - Since $\alpha \in S_n$, there is a unique $k$ such that $i = \alpha (k)$.
  - Since $\beta \in S_n$, there is a unique $j$ such that $k = \beta (j)$.

Commutativity

- Is it true that for every $\alpha, \beta \in S_n$, we have $\alpha \beta = \beta \alpha$?
- **No!**
  - Consider $S_3$ and
    \[
    \begin{array}{ccc}
    1 & 2 & 3 \\
    1 & 3 & 2 \\
    \end{array}
    \quad
    \begin{array}{ccc}
    1 & 2 & 3 \\
    2 & 1 & 3 \\
    \end{array}
    \]
    \[
    \alpha = \downarrow \quad \downarrow \quad \downarrow
    \beta = \downarrow \quad \downarrow \quad \downarrow.
    \]
    \[
    \begin{array}{ccc}
    1 & 2 & 3 \\
    3 & 1 & 2 \\
    \end{array}
    \quad
    \begin{array}{ccc}
    1 & 2 & 3 \\
    2 & 3 & 1 \\
    \end{array}
    \]
    \[
    \alpha \beta = \downarrow \quad \downarrow \quad \downarrow
    \beta \alpha = \downarrow \quad \downarrow \quad \downarrow.
    \]
### Associativity

- Is it true that for every $\alpha, \beta, \gamma \in S_n$, we have 
  
  $$(\alpha\beta)\gamma = \alpha(\beta\gamma)?$$

- **Yes.** In both cases, we get the function:

  \[
  \begin{array}{cccccc}
  1 & 2 & 3 & 4 & 5 \\
  \gamma: & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  2 & 4 & 5 & 1 & 3 \\
  \beta: & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  2 & 1 & 3 & 4 & 5 \\
  \alpha: & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  5 & 4 & 3 & 1 & 2 \\
  \end{array}
  \]

### The Identity Element of $S_n$

- **Identity.** The *identity permutation* is defined as $\text{id}(r) = r$ for every $r \in \mathbb{N}_n$. For any $\alpha \in S_n$, we have 
  
  $\text{id} \cdot \alpha = \alpha \cdot \text{id} = \alpha$. 

![Identity crisis.](image-url)
Inverse

• Is it true that for every $\alpha \in S_n$, there exists an inverse permutation $\alpha^{-1} \in S_n$ satisfying
  $\alpha \alpha^{-1} = \alpha^{-1} \alpha = \text{id}$.

• Yes. Just reverse the arrows.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
2 & 4 & 5 & 1 & 3 \\
\end{array}
\]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

Cycle Notation

• We can consider a permutation as a set of cycles.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
5 & 6 & 3 & 2 & 1 & 4 \\
\end{array}
\]

• We write this permutation as $(1 \ 5)(2 \ 6 \ 4)(3)$. 
Converting to Cycle Notation

- \( \alpha(1) = 2, \ \alpha(2) = 4, \ \alpha(3) = 5, \ \alpha(4) = 1, \ \alpha(5) = 3. \)

- We start with 1 and construct its cycle: \( 1 \rightarrow 2 \rightarrow 4 \rightarrow 1. \)

- We then choose a number that was not considered yet: \( 3 \rightarrow 5 \rightarrow 3. \)

- We got all the numbers of \( \mathbb{N}_5, \) so the cycle notation is \( (1 \ 2 \ 4)(3 \ 5). \)

Counting Cycles

**Problem.** How many distinct cycles of length \( k \) exist in \( S_n \)?

**Solution.**

- There are \( \binom{n}{k} \) ways of choosing \( k \) elements for the cycle.
- There are \( k! \) ways to order this elements.
- Each cycle has \( k \) different representations.

\[
\binom{n}{k} \frac{k!}{k} = \frac{n!}{k \cdot (n-k)!}.
\]
Card Shuffling

- **Problem.** Cards numbered 1 to 12 are picked up in row order and re-dealt in column order:

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<td>11</td>
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<td>4</td>
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How many times do we need to repeat this procedure until the cards return to their original positions?

Finding a Permutation

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- A reshuffling corresponds to a permutation.
- For example, after each reshuffling “6” moves to the position of “5”.

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<tbody>
<tr>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
</tbody>
</table>
Solution

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
10 & 11 & 12 \\
\end{array} \quad \begin{array}{ccc}
1 & 5 & 9 \\
2 & 6 & 10 \\
3 & 7 & 11 \\
4 & 8 & 12 \\
\end{array}
\]

- The **cycle structure** of the permutation: \( \alpha = (1)(2\ 5\ 6\ 10\ 4)(3\ 9\ 11\ 8\ 7)(12). \)
- Every cycle has length 1 or 5, so after **five** reshufflings we return to the original position.

Classification of Permutations

- The **type of a permutation** of \( S_n \) is the number of cycles of each length in its cycle structure.
- Both \((1\ 2\ 4)(3\ 5)\) and \((1\ 2\ 3)(4\ 5)\) are of the same type: one cycle of **length 3** and one of **length 2**.
  - We denote this type as \([2\ 3]\).
- In general, we write a type as \([a_1 a_2 a_3 a_4 ...]\).
Counting Permutations of a Given Type

**Problem.** How many permutations of $S_{14}$ are of the type $[2^23^24]$?

- We need to insert the numbers 1, 2, ..., 14 into the cycle pattern $(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)$.

- We can place every permutation of $\mathbb{N}_{14}$ into this pattern.
  - $(12\ 1)(3\ 5)(2\ 6\ 4)(13\ 14\ 3)(7\ 8\ 9\ 10)$
  - *Is the solution $14!$?*

Fixing the Solution

- The following permutations are identical:
  - $(12\ 1)(3\ 5)(2\ 6\ 4)(13\ 14\ 3)(7\ 8\ 9\ 10)$
  - $(3\ 5)(12\ 1)(2\ 6\ 4)(13\ 14\ 3)(7\ 8\ 9\ 10)$
  - So is the answer $\frac{14!}{2!2!}$?

- Another identical permutation:
  - $(1\ 12)(3\ 5)(2\ 6\ 4)(13\ 14\ 3)(7\ 8\ 9\ 10)$
  - So is the answer $\frac{14!}{2!2!2\cdot3\cdot3\cdot4}$?
  - *Yes!*
Counting Instances of a Type

In general, the number of permutations of $S_n$ of type $[1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} 4^{\alpha_4} \ldots]$ is

$$\frac{n!}{\alpha_1! \alpha_2! \alpha_3! \alpha_4! \ldots}$$

Types of $S_5$

<table>
<thead>
<tr>
<th>Type</th>
<th>Example</th>
<th>Number</th>
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<tbody>
<tr>
<td>$[1^5]$</td>
<td>id</td>
<td>1</td>
</tr>
<tr>
<td>$[1^3 2]$</td>
<td>(1 2)(3)(4)(5)</td>
<td>10</td>
</tr>
<tr>
<td>$[1^2 3]$</td>
<td>(1 2 3)(4)(5)</td>
<td>20</td>
</tr>
<tr>
<td>$[2^2]$</td>
<td>(1 2)(3 4)(5)</td>
<td>15</td>
</tr>
<tr>
<td>$[14]$</td>
<td>(1 2 3 4)(5)</td>
<td>30</td>
</tr>
<tr>
<td>$[23]$</td>
<td>(1 2 3)(4 5)</td>
<td>20</td>
</tr>
<tr>
<td>$[5]$</td>
<td>(1 2 3 4 5)</td>
<td>24</td>
</tr>
</tbody>
</table>
Conjugate Permutations

- Permutations $\alpha, \beta \in S_n$ are said to be conjugate if there exists $\sigma \in S_n$ such that $\sigma \alpha \sigma^{-1} = \beta$.

- Let $\alpha = (1\ 2)(3)$ and $\beta = (1)(3\ 2)$. The two permutations are conjugate, since we can take $\sigma = (1\ 2\ 3)$ and $\sigma^{-1} = (3\ 2\ 1)$.

\[
\begin{array}{c}
\sigma^{-1} \\
\downarrow & \downarrow & \downarrow \\
3 & 1 & 2 \\
\alpha \\
\downarrow & \downarrow & \downarrow \\
3 & 2 & 1 \\
\sigma \\
\downarrow & \downarrow & \downarrow \\
1 & 3 & 2
\end{array}
\]

Conjugation and Types

- **Theorem.** Two permutations of $S_n$ are conjugate iff they are of the same type.

$\alpha = (1\ 2)(3), \ \beta = (1)(3\ 2), \ \sigma = (1\ 2\ 3)$.

$\sigma \alpha \sigma^{-1} = \beta$
Proof: One Direction

- Suppose that $\alpha, \beta$ are conjugate, so there exists $\sigma$ such that $\sigma\alpha\sigma^{-1} = \beta$.
- Consider a cycle $\alpha(x_1) = x_2, \alpha(x_2) = x_3, \ldots, \alpha(x_k) = x_1$.
- Set $y_i = \sigma(x_i)$. Then $\beta(y_i) = \sigma\alpha\sigma^{-1}(\sigma(x_i)) = \sigma(x_{i+1}) = y_{i+1}$.

Proof: One Direction (cont.)

- $\sigma$ is a bijection between cycles of $\alpha$ and cycles of $\beta$.
- That is, $\alpha$ and $\beta$ are of the same type.
Proof: The Other Direction

- **Suppose \( \alpha \) and \( \beta \) have the same type.**
  - To prove conjugation, we need to find \( \sigma \).
  - Set up a bijection between the cycles of \( \alpha \) and \( \beta \), so that corresponding cycles have the same length.
  - For every two such cycles \( (x_1 \ x_2 \ \cdots \ x_k) \) and \( (y_1 \ y_2 \ \cdots \ y_k) \), we set \( \sigma(x_i) = y_i \). Then
    \[
    \sigma \alpha \sigma^{-1}(y_i) = \sigma \alpha(x_i) = \sigma(x_{i+1}) = y_{i+1} = \beta(y_i)
    \]
  - That is, \( \sigma \alpha \sigma^{-1} = \beta \).

The End

So how can we solve this?

In the next class... but you can try before that!