Ma/CS 6a

Class 5: Basic Counting

\[
\begin{array}{cccccc}
 & & & 1 & & \\
 & & 1 & & 1 & \\
 & 1 & & 2 & & 1 \\
1 & & 3 & & 3 & & 1 \\
 & 1 & 4 & 6 & 4 & 1 \\
 & 1 & 5 & 10 & 10 & 5 & 1 \\
 & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\end{array}
\]

By Adam Sheffer

- Send anonymous suggestions and complaints from here.
- Email: adamcandobetter@gmail.com
- Password: anonymous2

- There aren’t enough crocodiles in the presentations
- Why won’t you tell me how to solve the homework?!
- Only today! 75% off for Morphine and Xanax.
- Could you open every class by playing Flight of the Valkyries?
Permutations

- **Problem.** Given a set \( \{1, 2, \ldots, n\} \), in how many ways can we order it?
- **The case** \( n = 3 \). Six distinct orders / permutations: 123, 132, 213, 231, 312, 321.
- **The general case.**
  \[
  n! = n \cdot (n - 1) \cdot \cdots \cdot 2 \cdot 1
  \]

Total Number of Subsets

- **Problem.** How many subsets does the set \( S = \{1, 2, \ldots, n\} \) have?
  - Two options for every element \( i \in S \). Either \( i \) is in the subset or not.
  - Since there are \( n \) element in \( S \), the number of subsets is \( 2 \cdot 2 \cdot 2 \cdots 2 = 2^n \).
Subsets of Size $k$

- Given a set $\{1,2,\ldots,n\}$, how many (unordered) **subsets of size $k$** does it have?
- **Example.** Consider the case $n = 5$ and $k = 3$.
  - The possible subsets are $(1,2,3), (1,2,4), (1,2,5), (1,3,4), (1,3,5), (1,4,5), (2,3,4), (2,3,5), (2,4,5), (3,4,5)$.
  - 10 distinct subsets

Subsets of Size $k$ (cont.)

- Given a set $S = \{1,2,\ldots,n\}$, how many (unordered) subsets of size $k$ does it have?
- Look at the $n!$ orderings of $S$ and consider the first $k$ numbers as the subset.
  - For example, when $n = 5$ and $k = 3$
    - $12345$  $34251$
    - $13524$  $34152$
    - $54321$  $13542$
Binomial Coefficients

• Given a set $S = \{1, 2, \ldots, n\}$, how many (unordered) subsets of size $k$ does it have?
• Look at the $n!$ orderings of $S$ and consider the first $k$ numbers as the subset.
  ◦ Every subset is obtained $k! (n - k)!$ times, so

$$\binom{n}{k} = \frac{n!}{k! (n - k)!}$$

Pronounced “$n$ choose $k$”

Warm-up Problem

• **Prove or disprove.** For every $n \geq k \geq 0$

$$\binom{n}{k} = \binom{n}{n-k}.$$

• **True.** Deciding which $k$ elements to choose is like deciding which $n - k$ elements not to take.
Pascal’s Rule

- **Prove.** For every \( n \geq k \geq 0 \)

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.
\]

# of subsets containing 1  # of subsets not containing 1

Pascal’s Triangle

- **Pascal’s rule:** \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \).
- \( \binom{n}{k} \) is element \( k + 1 \) of row \( n + 1 \).

\[
\begin{array}{cccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\end{array}
\]

*Every number is the sum of the two numbers above it.*
A Sum of Binomial Coefficients

- **Prove.** For every $n \geq k \geq 0$

\[
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.
\]

- The left-hand side is the number of subsets of \{1,2,3,...,n\}, which is $2^n$.

Partitioning into $k$ Subsets

- **Problem.** For $n, k > 0$, we have $n$ identical balls and $k$ bins. In how many ways can we place the balls in the bins?

- **Example.** If we have three balls and two bins, there are four options: (3,0), (2,1), (1,2), (0,3).
Partitioning into \( k \) Subsets

- **Problem.** For \( n, k > 0 \), we have \( n \) identical balls and \( k \) bins. In how many ways can we place the balls in the bins?

- **Answer.** \( \binom{n + k - 1}{k - 1} \). The \( k - 1 \) choices correspond to the end of each bin.

\[
\begin{array}{ccccccccc}
\text{Bin #1:} & \text{Bin #2:} & \text{Bin #4:} \\
1 \text{ ball} & 3 \text{ balls} & \text{empty} \\
\end{array}
\]

The Binomial Theorem

- **Recall.**
  - \((x + y)^2 = x^2 + 2xy + y^2\).
  - \((x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3\).

- **The binomial theorem.** What is \((x + y)^n\)?

\[
\sum_{0 \leq i,j \leq n \atop i + j = n} \binom{n}{i} x^i y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \ldots
\]
The Binomial Theorem – Proof

- The binomial theorem.

\[(x + y)^n = \sum_{0 \leq i \leq n} \binom{n}{i} x^i y^{n-i}.\]

- Proof. We have

\[(x + y)^n = (x + y)(x + y) \cdots (x + y).\]

- The coefficient of \(x^i y^{n-i}\) is the number of ways to choose \(x\) from \(i\) of the parentheses and \(y\) from the remaining ones.

- That is, the coefficient of \(x^i y^{n-i}\) is \(\binom{n}{i}\).

The Binomial Theorem and Pascal’s Triangle

\[
\begin{align*}
(x + y)^1 &= x + y \\
(x + y)^2 &= x^2 + 2xy + y^2 \\
(x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
&\quad \vdots \\
1 \\
1 &1 \\
1 &2 &1 \\
1 &3 &3 &1 \\
1 &4 &6 &4 &1 \\
1 &5 &10 &10 &5 &1 \\
1 &6 &15 &20 &15 &6 &1
\end{align*}
\]
Monomials and Degrees

- Polynomials are sums of monomials:
  \[ x^7 + 3x^2y^4z + 5x^3z^3 + \cdots \]
- The degree of a monomial is the sum of the powers of its variables.
  \[ \text{deg}(3x^2y^4z) = 2 + 4 + 1 = 7 \]
- The degree of a polynomial is the maximum of the degrees of its monomials
  \[ \text{deg}(x^5 + 3x^2y^4z + 5x^3z^3) = 7 \]

Number of Monomials

- **Problem.** How many distinct monomials can a polynomial of degree \( D \) in \( k \) variables have?
- **Answer.** Take \( k + 1 \) bins – one for every variable and one extra. Every placement of \( D \) balls in the bins corresponds to a monomial.

![Diagram of monomials](image)
Number of Monomials

- **Problem.** How many distinct monomials can a polynomial of degree $D$ in $k$ variables have?

- **Answer.** Take $k + 1$ bins – one for every variable and one extra. Every placement of $D$ balls in the bins corresponds to a monomial.

\[
\binom{D + k}{k}
\]

Returning to Lecture 3

- To prove **Fermat’s little theorem** we assumed, without proof, that for any prime $p$

\[(a + b)^p \equiv a^p + b^p \mod p.\]

- **Proof.** By the binomial theorem:

\[(a + b)^p = \binom{p}{0} a^p + \binom{p}{1} a^{p-1} b + \binom{p}{2} a^{p-2} b^2 + \ldots\]

- To prove the claim, it suffices to prove that $p | \binom{p}{i}$ for every $1 \leq i \leq p - 1$.

- This holds since in $\binom{p}{i} = \frac{p!}{i!(p-i)!}$ the numerator is divisible by $p$ but the denominator is not.
Partitions of an Integer

- \( r, n \) – two positive integers.
- **Problem.** What is the number of solutions of
  \[
a_1 + a_2 + \cdots + a_r = n,
\]
  where each \( a_i \) is a natural number?

\[
5 = 1 + 1 + 3 = 1 + 3 + 1 = 0 + 0 + 5 = 1 + 0 + 4 = \cdots
\]

Solution

- Consider \( n \) as a sum of \( n \) unit elements.
- Dividing these elements across the \( r \) variables \( a_i \) is equivalent to placing \( n \) balls in \( r \) bins.
  - The value of \( a_i \) is the number of balls in the \( i^{th} \) bin.

\[
\binom{n + r - 1}{r - 1}
\]
Another Inequality

- **Problem.** Prove the identity

\[
\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.
\]

- **Proof.**
  - We begin with the identity
    \[
    (1 + x)^n(1 + x)^n = (1 + x)^{2n}.
    \]
  - By the **binomial theorem**, we have
    \[
    \left(\binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n\right)\left(\binom{n}{0} + \binom{n}{1}x + \cdots \right)
    \]

**Proof (cont.)**

\[
\left(\binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n\right)\left(\binom{n}{0} + \binom{n}{1}x + \cdots \right)
\]
Summing Up

- In how many ways can we choose \( k \) elements from \( \{1,2,3, ..., n\} \)?

<table>
<thead>
<tr>
<th></th>
<th>Ordered</th>
<th>Unordered</th>
</tr>
</thead>
<tbody>
<tr>
<td>No repetitions</td>
<td>( \frac{n!}{(n-k)!} )</td>
<td>( \binom{n}{k} )</td>
</tr>
<tr>
<td>With repetitions</td>
<td>( n^k )</td>
<td>( \binom{k+n-1}{n-1} )</td>
</tr>
</tbody>
</table>

Summing Up #2

- In how many ways can we place \( k \) balls into \( n \) bins?

<table>
<thead>
<tr>
<th></th>
<th>At most 1 ball in each bin</th>
<th>Any number of balls in each bin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Each ball has a different color</td>
<td>( \frac{n!}{(n-k)!} )</td>
<td>( n^k )</td>
</tr>
<tr>
<td>Balls are indistinguishable</td>
<td>( \binom{n}{k} )</td>
<td>( \binom{k+n-1}{n-1} )</td>
</tr>
</tbody>
</table>
The End

Imagine that you're drawing at random from an urn containing fifteen balls — six red and nine black.

OK, I reach in and... ...my grandfather's ashes??!! Oh God!

I... what?

Why would you do this to me??!