Homework 04 - Ma 5a - Solutions

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November 12, 2016

2.(b) (5 points) For any $g \in G$ and any $xyx^{-1}y^{-1} \in [G, G]$, we have
\[
     gxyx^{-1}y^{-1} = (gxy^{-1})(gy^{-1})(gx^{-1}y)(gyx^{-1}y^{-1})
\]
so $g[G, G]g^{-1} \subset [G, G]$ for all $g \in G$. Thus $[G, G]$ is a normal subgroup of $G$.

2.(c) (5 points) $G/N$ is abelian $\iff g_1g_2g_1^{-1}g_2^{-1}N = g_1Ng_2g_1^{-1}Ng_2^{-1}N = g_1g_1^{-1}g_2g_2^{-1}N = N$ for any $g_1, g_2 \in G \iff g_1g_2g_1^{-1}g_2^{-1} \in N$ for any $g_1, g_2 \in G \iff [G, G] \subset N$.

3.(a) (5 points) Let $G = S_3, H = \langle (1 2) \rangle, g = (1 3)$, then $g \notin H, [G : H] = 3, o(g) = 2$.

3.(b) (5 points) Suppose $H$ is normal in $G$ and $o(g)$ is relatively prime to $[G : H] = [G/H]$. Then $(gH)^{o(g)} = g^{o(g)}H = H$, so $o(gH)|o(g)$; also, since $gH \in G/H$, Lagrange’s Theorem implies $o(gH)|[G : H]$. Thus we must have $o(gH) = 1$, i.e. $gH = H$, so $g \in H$.

5.(b) (5 points) $\phi(\langle S \rangle) = \langle \phi(s) \rangle$ is straightforward because $x \in \phi(\langle S \rangle) \iff x = \phi(s_1 \cdots s_n)$ for some $s_i \in S$ (the $s_i$’s are not necessarily distinct) $\iff x = \phi(s_1) \cdots \phi(s_n)$ for some $s_i \in S \iff x \in \langle \phi(s) \rangle$.

For any $x \in \langle S \rangle Ker(\phi)$, we can write $x = sk$ where $s \in \langle S \rangle, k \in Ker(\phi)$, so $\phi(x) = \phi(sk) = \phi(s)\phi(k) = \phi(s) \in \phi(\langle S \rangle) = \langle \phi(s) \rangle$, so $\langle S \rangle Ker(\phi) \subset \phi^{-1}(\langle \phi(S) \rangle)$.

For any $x \in \phi^{-1}(\langle \phi(S) \rangle)$, there exist $s_i \in S$ not necessarily distinct such that $s = \phi^{-1}(\phi(s_1) \cdots \phi(s_n)) = \phi^{-1} \circ \phi(s_1) \cdots \phi^{-1} \circ \phi(s_n)$. Note that every $\phi^{-1} \circ \phi(s_i)$ is well-defined up to an element in Ker($\phi$), so $x \in \langle S \cup Ker(\phi) \rangle$.

Finally, for any $x \in \langle S \cup Ker(\phi) \rangle$, we can write $x = s_1k_1 \cdots s_nk_n$ for $s_i \in S, k_i \in Ker(\phi)$. Since Ker($\phi$) is a normal subgroup of $G$, any product of the form $k_j s_i$ can be written as $s_j k'_j$ for some $k'_j \in Ker(\phi)$, so after rearrangement we have $x = sk$ for some $s \in \langle S \rangle, k \in Ker(\phi)$. Thus $x \in \langle S \cup Ker(\phi) \rangle \subset \langle S \rangle Ker(\phi)$.

Concatenating the three inclusions proved above, we conclude $\langle S \rangle Ker(\phi) = \phi^{-1}(\langle \phi(S) \rangle) = \langle S \cup Ker(\phi) \rangle$. By a similar reasoning we obtain the same statement for Ker($\phi$)$\langle S \rangle$, so we conclude the desired equality.
6. (a) (5 points) By a previous homework problem we have \([G : H \cap K] = [G : H][H : H \cap K] = n[H : H \cap K]\), so \(n([G : H \cap K])\); similarly we obtain \(m([G : H \cap K])\). Thus \(lcm(m, n)\)\([G : H \cap K]\), and in particular \(lcm(m, n) \leq [G : H \cap K]\). For the other inequality, it suffices to show \([H : H \cap K] \leq [G : H \cap K]\). We do this by showing \(h(H \cap K) \neq h'(H \cap K) \Rightarrow hK \neq h'K\). Indeed, \(h(H \cap K) \neq h'(H \cap K) \Rightarrow h' - 1 h \notin H \cap K \Rightarrow h' - 1 h \notin K \Rightarrow hK \neq h'K\).

7. (a) (5 points) Suppose towards a contradiction that \(\mathbb{Q}_+^\times = \langle m/n \rangle\) for some rational number \(m/n\) with \(m, n\) coprime. Let \(p\) be an integer prime, then either \(p = m^k/n^k\) or \(p = n^k/m^k\) for some \(k\). In the first case we must have \(p|m\) and in the second case we must have \(p|n\), both of which cannot happen when \(p > m, n\). Thus \(\mathbb{Q}_+^\times\) is not cyclic.