1. (NH)
   
   (a) Recall that we saw in class that if a group $G$ is solvable, then any subgroup of $G$ and any quotient group of $G$ are solvable as well. Furthermore, if a group $G$ admits a normal subgroup $N$ such that $N$ and $G/N$ are solvable, then $G$ is solvable. Understand the proof of these facts if the sketch of proof in the class did not satisfy you. Of course, the trivial group is solvable. The following items use only those formal properties, so you don’t actually need to know what ”solvable” is. Thus, for example, you can replace ”solvable” by ”torsion” and get the same kind of structures and claims.
   
   (b) Let $G$ be a group and $H, N \subset G$ subgroups, and $N$ normal in $G$. Show that if $H$ and $N$ are solvable, then $HN$ is solvable.
   
   (c) Let $G$ be a finite group. Show that $G$ contains a unique solvable normal subgroup $R(G)$ with the property that any solvable normal subgroup $N$ of $G$ is contained in $R(G)$.
   
   (d) Let $G$ be a finite group. Show that $R(G/R(G)) = \{e\}$.

2. (NH)

   (a) Show that a group of order $455 = 5 \cdot 7 \cdot 13$ is abelian. Hint: If a group has an abelian quotient group, then the commutator subgroup is contained in the kernel of the quotient map. So if a group has ”a lot” of abelian quotients, we can hope to deduce
that the commutator subgroup is trivial, so that the group itself is abelian.

(b) Write a list of all the abelian groups of order \(2^4 \cdot 7\) up to isomorphism (i.e. every abelian group of that order should be isomorphic to some group in the list, and no two groups in the list should be isomorphic).

3. (NH) Let \(G = \mathbb{Z}/360\mathbb{Z} \oplus \mathbb{Z}/150\mathbb{Z} \oplus \mathbb{Z}/75\mathbb{Z}\). Find how many elements of order 5 are there in \(G\). Also, find how many subgroups of order 25 are there in \(G\).

4. (NH) Prove that any group of order 48 has a normal subgroup of order 8 or 16.

5. (NH) Let \(G\) be a non-abelian group of order 8. Show that \(|Z(G)| = 2\) and that if \(x \in G\) is of order 4, then the conjugacy class of \(x\) consists of 2 elements.

6. (NH) Let \(G\) be a group of order \(p^n\) (where \(p\) is prime and \(n \in \mathbb{Z}_{\geq 1}\)). Prove that \(|Z(G)| \neq p^{n-1}\).

7. (NH) Let \(G\) be a group of order \(p^n\), where \(p\) is prime and \(n \in \mathbb{Z}_{\geq 3}\). Find the number of conjugacy classes in \(G\).

8. (NH) Let \(G\) be a finite group acting on a finite set \(X\). Suppose that the action is transitive, and that \(|X| > 1\). Show that there exists \(g \in G\) such that \(gx \neq x\) for all \(x \in X\).

9. (NH) Let \(p\) be a prime. Find a \(p\)-Sylow subgroup in \(S_p\), and the number of \(p\)-Sylow subgroups in \(S_p\).

10. (NH)

(a) Let \(G\) be a group and \(H \subset G\) a subgroup. Show that there is a natural monomorphism from \(N_G(H)/C_G(H)\) to \(Aut(H)\) (induced by conjugation...).

(b) Let \(H = \langle (1234) \rangle \subset S_4\). Show that \(C(H) = H\) and \(N(H)\) is a 2-Sylow subgroup of \(S_4\).
(c) Let $F$ be a field, and $G = GL_n(F)$. Let $T \subset G$ be the subgroup of diagonal matrices. Show that $C_G(T) = T$ and $N_G(T)/C_G(T) \cong S_n$.

11. **(NH)** Prove that any group of order 105 is not simple.

12. **(NH)** Prove that there are no non-abelian simple groups of order $< 60$.

13. **(NH)** Let $G$ be a finite group, $p$ a prime, and $P \in Syl_p(G)$. Show that if $x \in N_G(P)$ has order $p^k$ for some $k \in \mathbb{Z}_{\geq 0}$, then $x \in P$.

14. **(NH)** Show that any group of order $5^2 \cdot 7^2$ has a subgroup of order $5^2 \cdot 7$.

15. **(NH)** Let $F$ be a field. Let $G = GL_n(F)$. Let $B \subset G$ be the subgroup consisting of upper-triangular matrices. Show that $B$ is solvable.

16. **(NH)** Let $p < q < r$ be primes. Show that any group of order $p$, $pq$ or $pqr$ is solvable.